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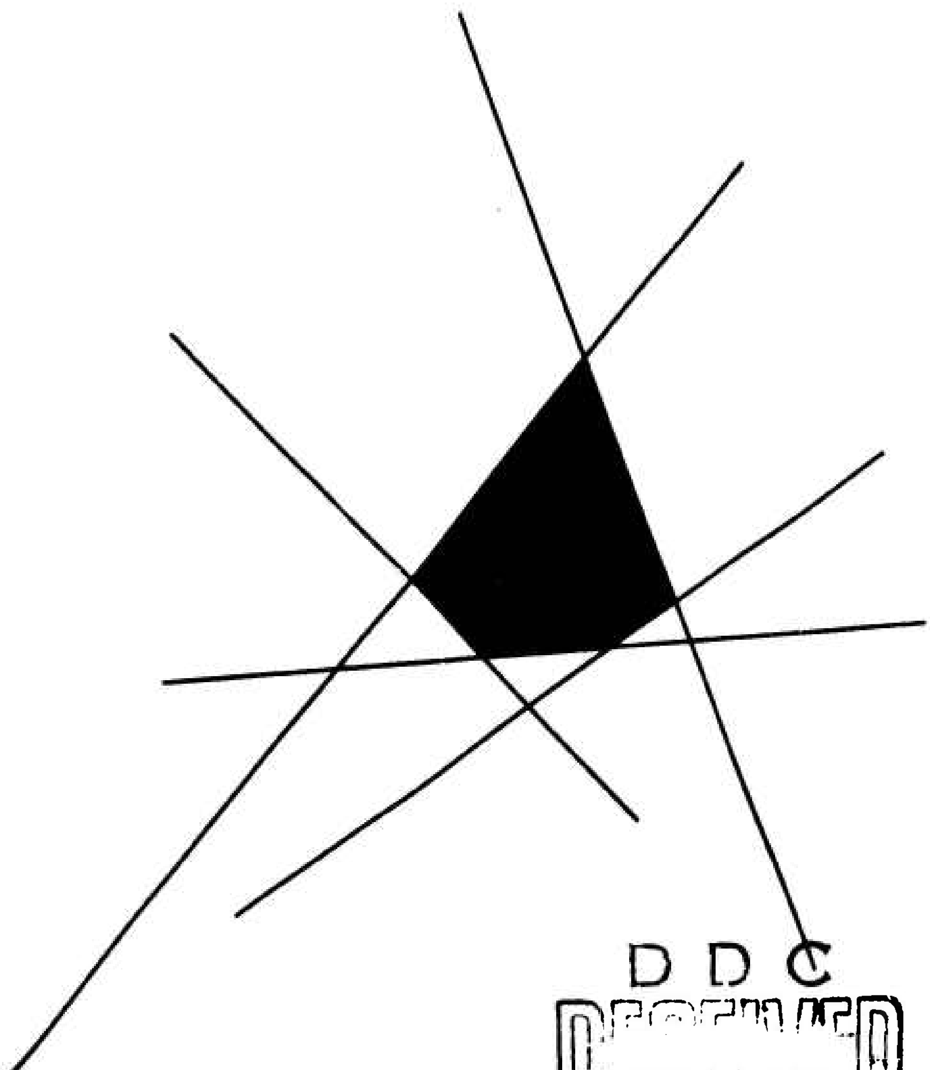
MATHEMATICAL PROGRAMMING AND OPTIMAL CONTROL THEORY

by

RICHARD M. VAN SLYKE



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136

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AND OPTIMAL CONTROL THEORY

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L = (P = (P sub 0, ..., P sub m)): P sub 1 = P sub 2 = ... = P sub m = 0)

ABSTRACT

E superscript (m+1)

Let K be a closed convex set in E^{m+1} and $L = \{P = (P_0, \ldots, P_m) : P_1 = P_2 = \ldots = P_m = 0\}$. Then for the simple problem:

Minimize P_0 P sub 0

Subject to $P = (P_0, P_1, \ldots, P_m) \in K \cap L$,

P = (P sub 0, P sub 1, ..., P sub m) epsilon the intersection of K and L;

we prove a duality theorem and the convergence of a solution algorithm modeled on the duality theorem and the simplex method of linear programming respectively.

Specialization of this general model to linear programming, convex programming, generalized programming, control theory, and the decomposition approach to mathematical programming yield the appropriate duality theorems and solution algorithms in each case. ( )

The principle idea exploited here is the notion of supporting hyperplanes to convex sets. The duality theorem is a direct application of the fact that every boundary point of a convex set belongs to a supporting hyperplane; moreover, the generalized simplex method presented here is most useful when K is characterized in such a way that, given a hyperplane, the translate of it which is a supporting hyperplane of K may easily be found, if it exists, as well as the points of the supporting hyperplane which are common to the boundary of K. This corresponds to "pricing out" in linear programming.

We show that many problems in control theory are special cases of our model and for a large class of linear control problems a solution method is outlined to illustrate the use of mathematical programming techniques to solve optimal control problems.

Two appendices contain the elements of the theory of affine spaces, of convex sets, and of ordinary differential equations used in the text.

TABLE OF CONTENTS

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Introduction

Since the end of World War II, many theories for the "optimization" of mathematical systems have been developed, among them: mathematical programming, game theory, decision theory, optimal control theory, and dynamic programming. Concurrently an increased interest in the more classical calculus of variation has taken place. Since all these methods are concerned with optimization problems, it is natural to seek a unified theory of mathematical optimization. Any

consolidation and abstraction of mathematical theories is based on a few unifying ideas. This report is based on two ideas which I heartily endorse as candidates for use in the foundation of a general theory. The first is the supporting hyper-plane theorem for convex sets. In disguise, this appears, for example, in the duality theory of mathematical programming, in the maximum principle in control theory, and as one player's strategy in the theory of matrix games. The second idea arises from the emphasis on methods of solution and constructive proofs more or less inherent in the very nature of optimization problems. This idea is the simplex method of G. B. Dantzig which so far is the most successful solution method for a wide class of optimization problems. Just why it is as efficient as it is, is still a tantalizing question,* but to date, the simplex method is the best method for solving linear programs, and matrix games and moreover,

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*Probably the only unsolved question of theoretical interest in linear programming. Essentially, if a linear program has n columns and m rows, the solution consists of searching through various bases to find one which yields the optimal solution. Using the upper bound on the number of bases obtained by assuming that every collection of m columns may form a basis, the number of possible bases might be expected to be of the form $\binom{n}{m}$ the number of ways n objects can be chosen m at a time. Experience has shown that the number of iterations is of the smaller order of m to 3m, and no convincing explanation of this curious fact has been found. In 1957, W. M Hirsch [LPE, p.160] conjectured that it is possible to progress from one feasible basic solution to any other feasible basic solution in m steps each intermediate basic solution also being feasible. To date, the conjecture has been verified for $m < 5$ with the feasibility set bounded. If the feasibility set is allowed to be unbounded Klee and Walkup [10a] have demonstrated that the conjecture is false for $m \geq 4$.

forms an integral part of many non-linear programming schemes. Its use in the theory of optimal control has been unnecessarily limited, but part of the purpose of this report is to illustrate the application of the simplex method to optimal control problems.

The basis of this report is the following very simple problem: Suppose $\mathcal{X}$ is a closed convex subset of euclidean m+1 space, E^{m+1}, and $\mathcal{L}$ is the line $\mathcal{L} = \{(P_0,\ldots,P_m) / P_1 = P_2 = \ldots = P_m = 0\}$; we wish to minimize P_0 over the set of all points $P = (P_0,\ldots,P_m) \in \mathcal{L} \cap \mathcal{X}$. Clearly the optimizing point P^* is a boundary point of $\mathcal{X}$ and is unique if it exists. By a well known theorem in the theory of convex sets there exists a "supporting" hyperplane, $\mathcal{H}$, in E^{n+1} with the properties that P^* belongs to the hyperplane and $\mathcal{X}$ is contained entirely in one of the closed half spaces determined by the hyperplane. $\mathcal{H}$ can be expressed in terms of its normal, $\pi = (\pi_0,\ldots,\pi_m)$, as $\mathcal{H} = \{P / \pi P + \tilde{\mu} = 0\}$ for some scalar $\tilde{\mu}$. We remark in passing that if $\mathcal{X}$ is determined by a differentiable surface, then π is proportional to the gradient of the surface at P^*.

Another way of expressing the above problem is

(1) Maximize $\quad Z$

subject to $\quad U_0 Z + P = 0$

where $U_0 = (1,0,\ \ldots,\ 0)^T$.

The fundamental problem (1) can be reformulated in an equivalent form which will be of great utility for us. This formulation is:

$$
\text{(2)} \qquad \begin{aligned} &\text{Maximize} \quad Z \\ &\text{subject to} \quad U_0 Z + \sum \lambda_j P^j = 0 \\ &\qquad\qquad\qquad \sum \lambda_j = 1 \\ &\qquad\qquad\qquad (\lambda_j \geqq 0) \end{aligned}
$$

where again each P^j may be chosen freely from $\mathcal{K}$. The equivalence follows from the convexity of $\mathcal{K}$.

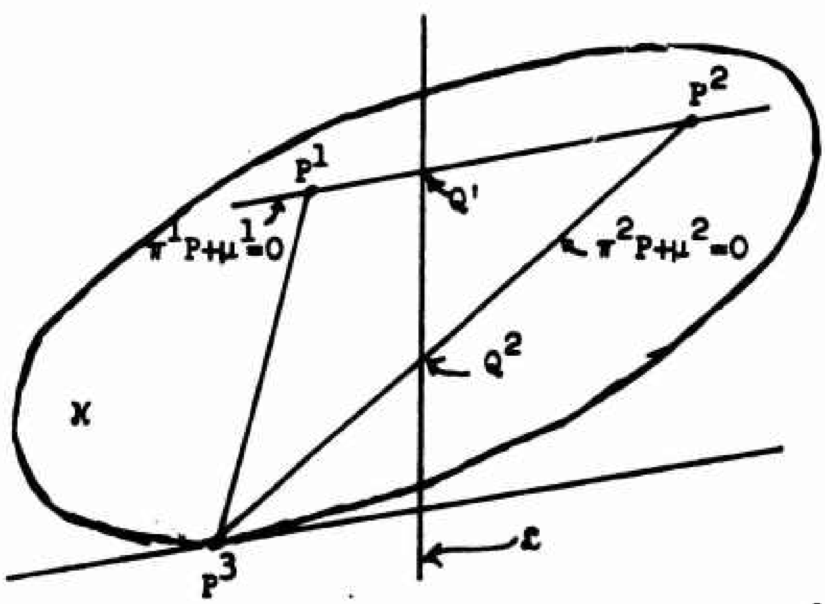
We then take the formal dual of the above problem in the sense of linear programming, obtaining:

$$
\text{(3)} \qquad \begin{aligned} &\text{Minimize} \quad \mu \\ &\text{subject to} \quad \pi P + \mu \geqq 0 \quad \text{for all} \quad P \in \mathcal{K}. \\ &\qquad\qquad\qquad \pi U_0 = \pi_0 = 1. \end{aligned}
$$

If $\tilde{\pi}_0 \neq 0$, for some supporting hyperplane of $\mathcal{K}$ at P^* it is easy to see that $\pi = \frac{1}{\tilde{\pi}_0} \tilde{\pi}$ and $\mu = \frac{1}{\tilde{\mu}_0} \tilde{\mu}$ is an optimal solution of the dual problem, and of course in the other direction, solutions of (3) determine supporting hyperplanes of $\mathcal{K}$ at P^*.

$\mathcal{K}$ can be characterized in many ways depending on the application to which our model is applied. In many cases it is much easier to find boundary points of convex sets

corresponding to a given normal π, than to find directly the "lowest" point in $\mathcal{X}$ on the line $\mathcal{L}$. In the context of the simplex method for linear programming, this simply expresses the fact that "pricing out" is easier than directly finding the solution. When this property is present, it is natural to generalize the simplex method to solve the fundamental problem. To illustrate the basic idea we initially restrict ourselves to the case where $\mathcal{X}$, in addition to being closed and convex, is bounded. We assume we start out with points $P^1, \ldots, P^{m+1} \in \mathcal{X}$ which uniquely determine a hyperplane in E^{m+1}, which is not parallel to $\mathcal{L}$, and such that some point $Q^1 \in \mathcal{L}$ is in the convex hull of $P^1, \ldots, P^{m+1}$.



We then move the hyperplane determined by $P^1, \ldots, P^{m+1}$ in the direction of decreasing P_0 until we find the corresponding boundary point, P^{m+2}. If π^1 is the unique vector

determined by $\pi^1 P^1 = \pi^1 P^2 = \ldots = \pi^1 P^{m+1}$, then finding P^{m+2} can be expressed in the following way:

(4) $\quad \pi^1 P^{m+2} = \mathrm{Min}\ \{\pi^1 P / P \epsilon \mathcal{K}\}$.

This we call "pricing out", or solving the "subproblem" in various contexts.

The next step is to solve (2) for these particular P^j, $j = 1,\ldots,m+2$ no longer allowing them to vary over $\mathcal{K}$. (2) is then just a linear program which we can solve obtaining an optimal solution over the convex hull of $P^1,\ldots,P^{m+2}$. We call this solution Q^2 and the corresponding dual variables, π^2, which are determined by the m points of $P^1,\ldots,P^{m+2}$ which are in the optimal basis. π^2 is then used in the "subproblem" (4) to generate a new point P^{m+3}. We repeat the process, until $\mathrm{Min}\ \{\pi^k P / P \epsilon \mathcal{K}\} = \mathrm{Min}\ \{\pi^k P^j,\ j = 1,\ldots m+k\}$.*

We now consider three of the assumptions we have made so far. The first assumption, and most important, is that there is a supporting hyperplane at P^* determined by $\tilde{\pi},\tilde{\mu}$ with $\tilde{\pi} \neq 0$. That is, a supporting hyperplane not containing $\mathcal{L}$. If $\mathcal{K}$ is a polyhedral set, i.e., the intersection of a finite number of half-spaces, we have no trouble since polyhedral sets are "pointed" in the sense that the cone of normals to supporting hyperplanes at an extreme point has a non-empty interior. For the general case when $\mathcal{K}$ is not necessarily polyhedral, we assume that the intersection

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*Barr has developed a similar technique [1a] in the case where (2) is a quadratic program.

of K and $\mathcal{L}$ contains a point in the relative interior of K. If K and $\mathcal{L}$ have this property, we say that (1) is <u>regular</u>.* The next assumption we consider is that we can find the points $P^1, \ldots, P^{m+1}$ to start our algorithm. If K has a non-empty interior, regularity implies these points exist. If K has no interior, then the whole problem can be considered in a lower dimensional space where it has an interior. The practical problem of finding the points is solved by using a "phase I" type procedure as in linear programming. The "phase I" process also indicates the dimensionality of K. This is discussed at the end of Chapter II. The last assumption we discuss is the assumption that K is bounded. For a regular problem with K bounded, the algorithm outlined above was shown to converge to solutions of (1) and (3) by G. B. Dantzig in the context of convex programming [L.P.E., Ch. 24]; however, in order to include most algorithms of mathematical programming of the simplex-type in this format, it is necessary to allow K to be unbounded. In particular, for the linear program

$$\begin{array}{ll} \text{Maximize} & Z \\ \text{subject to} & U_0 Z + Ax = b \\ & x \geqq 0 \end{array}$$

we have $K = \{P/P = Ax - b,\ x \geqq 0\}$, which is a cone and patently unbounded. A primary objective then is to extend the algorithm to the unbounded case. After this is done in Chapter II, we show that linear programming, convex

*Much of the duality theory used here has also been developed by T. Rockafellar using the theory of conjugate functions [15a]; see also [17a]. Regularity, as used here, is related to Rockafellar's notion of a <u>stably set</u> problem.

programming, and the decomposition algorithm of mathematical programming are all special cases.

Other special cases are problems in control theory. In Chapter III we first establish a very general class of problems which can be formulated as special cases of (1). Suppose we have a system of ordinary differential equations in vector form.

$$(5) \quad \frac{dx(t)}{dt} = f(x(t),u(t),t), \quad x(t) = (x_0(t),\ldots,x_n(t)) \in E^{n+1}$$

$$u \in \Omega, \qquad u(t) \in E^r$$

$$x(0) \in X^0 \qquad t \in [0,\infty].$$

For any time $t > 0$, we can define $S_t(x^0)$ to be the set of points that can be reached at time t by the solution of (5) for some point $x(0)$ in X^0 and u in the class of "admissible controls" Ω.

For the special case where $\Omega = \{u \,/\, u(t) \in Q(t),\ u(t)$ measurable and bounded$\}$ where $Q(t)$ is a compact convex set for all t, $0 \leq t \leq T$ and as a function of t is upper semi-continuous, and $f(x,u,t)$ is given by $f_i(x,u,t) = \sum_{j=1}^{m} A_i^j(t)\, x_j(t) + u(t)$, $i = 1,\ldots,m$ and $f_0(x,u,t)$ is convex in (x,u), where each A_i^j is continuous on $0 \leq t \leq T$, we show that the set $S_T^+(S^0) = \{x = (x_0,\ldots,x_n) \,/\, x_0 \geq \overline{x}_0,\ x_1 = \overline{x}_1,\ldots,x_n = \overline{x}_n,\ \overline{x} \in S_T(X^0)\}$ is closed, convex and "bounded below."

For the optimal control problem of minimizing $x_0(T)$ over solutions of (5) which satisfy $x_i(T) = x_i^T$, $i = 1,\ldots,n$ we may use the generalized simplex method if the above conditions are satisfied and $\mathcal{X} = \{P / P = x - (0, x_1^T, \ldots, x_n^T),\ x \in S_T^+(X_0)\}$ is regular.

In practice, the "pricing out" operation is by no means simple, but in the case where $f(x,u,t) = A(t)\,x(t) + u(t)$ life becomes much simpler. We close the thesis by outlining solution methods in this case. One approach is to discretize the problem in time and to solve the corresponding linear program. This is expedited by the use of generalized programming and a new algorithm [5] which makes use of the special structure of the resulting linear program.

In Chapter I, the simplex method as applied to linear programming is outlined in its algebraic form, mainly to get notation straight. We then give the formal details of the generalized programming approach of Wolfe and Dantzig [L.P.E., Ch. 22]. The details and relevant proofs are deferred to Chapter II. This generalized simplex method forms the computational basis of this work. Finally, another variant of the simplex method, the generalized upper bounding technique [4,5], is reviewed, first as an illustration of some of the devices used to make the basic simplex method more efficient in special cases and, second, because of its utility in the numerical solution of certain types of linear optimal control problems.

In Chapter II the fundamental problem is introduced. We then prove the duality theorem for this problem and show how the generalized simplex method applied to this problem converges to the optimal solution. We then show that the ordinary simplex method, convex programming, generalized programming, the decomposition algorithm for linear programming, and many problems of control theory, are special cases of this problem and interpret the generalized simplex method in each of these cases.

In Chapter III the problem of optimal control is treated in more detail, first, in the spirit of Chapter II, and then more mundanely, by discretizing the problem and solving the resulting linear programming problem using the generalized upper bounding technique.

Finally there are several appendices giving necessary theorems from other fields of mathematics with adequate references to proofs if they are not given in the text.

A final note on references. Numbers in square brackets, [ ], will refer to the list of references at the end of this report. The special symbol [L.P.E.] refers to G. B. Dantzig's Linear Programming and Extensions [3].

/// denotes the end of a proof. Theorems, remarks, and equations are all numbered serially on the same scale. The number referring to the chapter is dropped if the reference is made in that chapter.

Chapter I: Some Ideas in the Theory of Mathematical Programming

0. Some Notation: Matrices will be denoted by capital letters, e.g., A. The i^{th} row of A will be denoted by A_i, the j^{th} column by A^j and the element in the i^{th} row and j^{th} column by A_i^j. The components of vectors are distinguished by subscripts. Superscripts on vectors represent an enumeration of a collection of vectors. Generally speaking, column vectors will be denoted by Latin letters and row vectors, by Greek. The norm $|| x ||$ of a vector x will be the euclidean norm $\left(\sum x_i^2\right)^{1/2}$ and the norm $|| A ||$ of a matrix will be the usual one

$$|| A || = \sup_{||x|| \leq 1} || Ax || \quad \text{where the norms in the right}$$

are euclidean. A symbol of the form $\{x/A\}$ denotes the set of all x possessing the property A.

1. Linear Programming and the Simplex Method: The fundamental problem of linear programming is:

$$
\text{(1.1)} \quad
\begin{aligned}
&\text{Maximize} \quad x_0 \\
&\text{Subject to} \quad A_0^0 x_0 + A_0^1 x_1 + \cdots + A_0^n x_n = b_0 \\
&\qquad\qquad\quad A_1^0 x_0 + A_1^1 x_1 + \cdots + A_1^n x_n = b_1 \\
&\qquad\qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\
&\qquad\qquad\quad A_m^0 x_0 + A_m^1 x_1 + \cdots + A_m^n x_n = b_m \\
&\text{and} \qquad\quad x_1, \ldots, x_n \geq 0.
\end{aligned}
$$

In terms of the columns A^j of $A(= [A_i^j])$ an equivalent formulation of (1.1) is

Maximize x_0

subject to

$$
\text{(1.2)} \qquad \sum_{j=0}^{n} x_j A^j = b
$$

$$
x_1, \ldots, x_n \geq 0
$$

Associated with the fundamental problem* is the dual problem:

---

* In this context, the fundamental problem is often referred to as the primal problem.

$$
\begin{array}{llll}
\text{Minimize} & b_0\pi_0 + b_1\pi_1 + \cdots + b_m\pi_m & & \\
\text{subject to} & A_0^0\pi_0 + A_1^0\pi_1 + \cdots + A_m^0\pi_m & = & 1 \\
(1.3) & A_0^1\pi_0 + A_1^1\pi_1 + \cdots + A_m^1\pi_m & \geq & 0 \\
 & \vdots & & \\
 & A_0^n\pi_0 + A_1^n\pi_1 + \cdots + A_m^n\pi_m & \geq & C
\end{array}
$$

or equivalently

$$
\begin{array}{lll}
 & \text{Minimize} & \pi b \\
(1.4) & \text{subject to} & \pi A^0 = 1 \\
 & & \pi A^j \geq 0 \quad j = 1, \ldots, n
\end{array}
$$

A vector $x = x^0$ is called a <u>feasible solution</u> of (1.1) or (1.2) if $x_1^0, \ldots, x_n^0 \geq 0$ and $\sum x_j^0 A^j = b$. x^0 is an <u>optimal</u> solution if x_0^0 is maximal over all x which are feasible. Similar terminology applies to the dual program.

(1.2) and (1.4) (or equivalently (1.1) and (1.3)) are related by the following theorems:

1.5 <u>Theorem</u> (Duality): <u>If feasible solutions to both (1.2) and (1.4) exist, then the value, x_0, corresponding to any feasible solution of the primal (1.2) is less than or equal to the value πb for any feasible solution of the dual (1.4). Furthermore, optimal solutions exist for both systems and they are equal to a common value, i.e., Max x^0 = Min πb.</u>

1.6 Theorem (Unbounded Solutions): (a) If the primal (1.2) has a feasible solution and the dual (1.4) does not, then x_0 is not bounded above over the set of feasible solutions to the primal, i.e., there is no optimal solution to the primal. (b) If the dual (1.4) has a feasible solution and the primal (1.2) does not, then πb is not bounded below over the set of feasible solutions to the dual.

1.7 Theorem (Optimality Criterion): If x^0 and π^0 are feasible solutions to the primal (1.2) and dual (1.4) systems respectively and if we let: $\Delta_j^0 = \pi A^j$ then x^0 and π^0 are respectively optimal solutions to the primal and dual problems iff* $\Delta_j^0 x_j^0 = 0 \quad j = 1, \ldots, n.$

---

*iff = if and only if

The proofs of these theorems can be found in Chapter 6 of [L.P.E.].

The most common method for determining optimal solutions is the simplex method or one of its variants. As a review and in order to build up a consistent notation, we sketch one of the variants here.

We assume in (1.2) that A is of rank m + 1. A basis for (1.2) is a set of m + 1 linearly independent columns $A^0, A^{j_1}, \ldots, A^{j_m}$ which we will require to contain A^0. We usually denote a basis by B where $B^0 = A^0$ and $B^1 = A^{j_1}$,

$i = 1,\ldots,m$. A <u>basic solution</u>, x^b, of (1.2) is a solution obtained from a basis $B = [A^0, A^{j_1}, \ldots, A^{j_m}]$ by letting

$$x^b_j = 0 \qquad j \neq 0 \qquad j \neq j_i \qquad i = 1,\ldots,m$$

$$x^b_{j_i} = x^*_i \qquad i = 1, \ldots, m \; ; \; x^b_0 = x^*_0 \; .$$

where x^* is the unique solution of $Bx^* = b$. Note that x^b is not necessarily nonnegative. A <u>basic feasible solution</u> is a feasible solution which is basic.

The following theorem allows us to consider only basic feasible solutions:

1.8 <u>Theorem</u>: <u>If there exists an optimal solution x^0 to (1.2), then there exists a basic optimal solution x^1.</u>

<u>Proof</u>: Let x be any optimal solution with $x_0 = x^0_0$ and let $J = \{ j / x_j > 0 \}$. Either $\{A^j\}_{j \in J}$ forms a linearly independent set or there exists y such that $\sum_{j \in J} y_j A^j = 0 \qquad (y_j = 0,\ j \notin J)$.

If the first occurs, we are done; suppose in fact, the second possibility occurs. If $0 \in J$, clearly $y_0 = 0$ for otherwise letting $Z(\lambda) = x + \lambda y$; Z_0 could be made greater than x_0 by appropriately choosing λ contradicting optimality. Suppose $j_0 \in J$, and without loss of generality that $y_{j_0} < 0$.

Then $Z(\lambda)$ must eventually have one less non-zero component than x as λ increases. By repeating this process a finite number of times, the columns with non-zero coefficients will eventually form a linearly independent set.///

Let us suppose we have an initial basic feasible solution x^0 and the corresponding basis

$$B = [A^0, A^{j_1}, \ldots, A^{j_m}].$$

B is, by definition, non-singular, so we may assume we also have B^{-1}, the inverse of B. Let the "prices", π, be given by $\pi = (B^{-1})_0$, i.e., the 0^{th} row of B^{-1}. Then we have $\pi A^0 = 1$ and $\pi A^{j_i} = 0$, $i = 1, \ldots, m$. If $\pi A^j \geq 0$ for all j we have by (1.7) optimal solutions to the primal and dual problems and we are done.

If not, determine s by

$$\pi A^s = \min \pi A^j < 0 \qquad j = 1, \ldots, n.$$

We now wish to form a new basis, B', by replacing one of the columns of the old basis B by A^s in such a way as to guarantee that the new basic solution x' corresponding to B' is feasible.

What we are doing by introducing A^s into the basis is increasing the value of x_s in the equation $Ax = b$ maintaining equality by adjusting the old basic variables. Let θ denote the value of x_s and multiply $Ax = b$ on the left by B^{-1} (and assume for notational simplicity that $B = [A^0, A^1, \ldots, A^m]$) obtaining:

$$(1.9)\quad x_0U_0 + x_1U_1 + \ldots + x_mU_m + x_{m+1}B^{-1}A^{m+1} + \ldots + \theta B^{-1}A^s + \ldots + x_nB^{-1}A^n = B^{-1}b$$

Where U_i is the unit vector with 1 in the i^{th} component. Then

$$(1.10)\quad x_i = (B^{-1}b)_i - \theta(B^{-1}A^s)_i \qquad i = 0, \ldots, m.$$

But $x_i \geq 0 \quad i = 1, \ldots, m$ must be satisfied so we can only increase θ up to

$$(1.11)\quad \theta = \frac{(B^{-1}b)_r}{(B^{-1}A^s)_r} = \min_{(B^{-1}A^s)_i > 0} \frac{(B^{-1}b)_i}{(B^{-1}A^s)_i} \quad \text{where}$$

$$(B^{-1}A^s)_r > 0.$$

Since x_{j_r} is now zero, A^{j_r} can be dropped from the basis. If there is no r such that $(B^{-1}A^s)_r > 0$ it can be shown that x_0 is unbounded above and increasing x_s gives a class of feasible solutions with arbitrarily large values of x_0. In texts on linear programming it is shown, if suitable care is taken for cases where $\theta = 0$ in (1.11), that this

process either yields an optimal solution or a class of feasible solutions which are unbounded above in x_0 in a finite number of steps [L.P.E., Chs. 6 and 10]. To get the initial basis the "Phase I" procedure is used. [L.P.E., Ch. 5]

1.12 Remark: It can be shown that any optimization of a linear form in a finite number of variables over a closed convex polyhedral set can be expressed as a problem of the form 1.2 and be solved using the simplex method. For example if we wish to

(1.13) minimize $\gamma y = \Sigma \gamma_j y_j$

subject to $y \epsilon K$ a convex polyhedral set, we first express K as the intersection of a finite number of half spaces (A.2.7):

$$\sum D_i^j y_j \leqq d_i \qquad i = 1, \ldots, m.$$

We then define

$$y_j^+, y_j^- \text{ by } y_j = y_j^+ - y_j^- \qquad y_j^+, y_j^- \geqq 0$$

and define "slack variables" by

$$S_i = d_i - \sum D_i^j y_j$$

$$S_i \geqq 0.$$

Since maximizing $-\gamma y$, is equivalent to minimizing γ_y

(1.17) Max x_0

$$\text{subject to } X_0 = -\sum{}^{1} \gamma_j \, (y_j^+ - y_j^-)$$

$$\text{and } \sum D_i^j \, (y_j^+ - y_j^-) + S_i = d_i \qquad i = 1,\ldots,m$$

$$y_j^+, y_j^- \geq 0$$

is equivalent to (1.13) and is of the form (1.1).

2. Generalized Programming: A very fruitful generalization of the fundamental problem is the following problem (due to P. Wolfe):

$$(2.1) \quad \max. \quad x_0$$

$$\text{s.t.} \quad \sum_{j=0}^{n} x_j A^j = b \quad \text{where each } A^j \text{ may be chosen}$$

freely from $\mathcal{O}_j (j \neq 0)$, a closed convex subset of $m + 1$ dimensional euclidean space E^{m+1}.

Formally we "solve" (2.1) by first choosing one representative A^{0j} of each set $\mathcal{O}_j$ and forming the corresponding fundamental linear program. Using the simplex algorithm, an optimal basic solution and the corresponding optimal dual solution π^0 are obtained. We next solve the n sub-problems:

$$(2.2) \quad \begin{array}{ll} \min & \pi^0 A^j \\ \text{s.t.} & A^j \in \mathcal{O}_j \qquad j = 1, \ldots, n \end{array}$$

To solve (2.2), let

$$(2.3) \qquad \Delta_j = \min_j \pi^0 A^j \quad \text{and} \quad \Delta = \min_j \min_{\mathcal{O}_j} \pi^0 A^j$$

and suppose $\Delta_s = \Delta$. If $\Delta \geq 0$ we stop and if $\Delta < 0$ we add A'^s (where $\pi^0 A'^s = \Delta$) to the master program and reoptimize obtaining a new $\pi = \pi^1$. We then repeat the process. We consider this generalized algorithm in more

detail in the next chapter, here we merely indicate the justification of this formal procedure in the important case where the $\mathcal{O}_j$ are polyhedral sets. In this case (2.2) is a linear program according to 1.12. The simplex method applied to the j^{th}-subproblem yields either an extreme point $A^j \epsilon \mathcal{O}_j$ as a minimum; or a class of vectors of the form $A^j + \lambda H^j$ with $\pi^0 H^j < 0$ and $A^j + \lambda H^j \epsilon \mathcal{O}_j$ for all $\lambda \geq 0$ and all $A^j \epsilon \mathcal{O}_j$. In the former case, we add A^j and in the latter, H^j as additional columns to (2.1), referred to as the master program. The A^j and H^j are extreme points and rays respectively and there are only a finite number of them (up to a normalization of the H^j). Since the simplex method is finite and the number of columns which can be added are finite, the algorithm for the generalized program with polyhedral coefficient sets is finite.* In the general case, the algorithm is not finite, but under very general conditions converges, which we will show later in Chapter II. Now we turn to a variant of the simplex method which will prove useful in Chapter III.

3. <u>Generalized Upper Bounding Techniques</u>: We shall have need to apply the simplex method in a shrewd way to special linear programs consisting of M + L equations such that each variable has at most one non-zero coefficient in the last L equations, and the last L equations are not homogeneous, i.e., the last L components of the right hand side are all non-zero. Dantzig suggested informally two efficient ways of attacking such

*Care has to be taken in the use of this procedure. We give the precise results in Section II.5.

problems, the one given here is set out in more detail in [5], the other is described in [4]. By normalizing the variables and multiplying the equations by constants we can assume without loss of generality that all the coefficients in the last L equations are ± 1. Moreover, for simplicity in describing the algorithm, we assume that these coefficients are all + 1. The obvious modifications required to handle -1 coefficients will be indicated later.

The <u>ℓ^{th} set of variables or columns</u>, S_ℓ, will refer (depending on context) to those variables or columns corresponding to the columns of coefficients with 1 as their $M + \ell^{th}$ component. S_0, <u>the zero set</u>, is the set corresponding to columns with zero's for the $M + 1^{st}$ to $M + L^{th}$ coefficients.

We assume that the system is of full rank and denote by $[\underline{A}^{j_1},\ldots,\underline{A}^{j_{M+L}}]$ a basis for the system. We always assume that $\underline{A}^{j_1} = \underline{A}^0$, where x_0 is the variable to be maximized.

The underscoring is to differentiate coefficient vectors with all M + L components from the reduced vectors consisting of the first M coefficients which will be denoted without the underscoring. There will be no underscoring for individual coefficients A_i^j of the two different types of vectors since they differ only in the <u>number</u> of components. The proofs of the following obvious theorems are in [4].

3.1 <u>Theorem</u>: <u>At least one variable from each set S_ℓ is basic</u>.

(3.3)

$$
\begin{array}{cccccccccccccc}
A_0^0 & \cdots & A_0^{\ell_0} & A^{\ell_0+1} & A_0^{\ell_0+2}-A_0^{\ell_0+1} & A_0^{\ell_0+3}-A_0^{\ell_0+1} & \cdots & A_0^{\ell_1}-A_0^{\ell_0+1} & \cdots & A_0^{\ell_{L-1}+1} & \cdots & A_0^{\ell_L}-A_0^{\ell_{L-1}+1} & = & b_1 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
A_M^0 & \cdots & A_M^{\ell_0} & A_M^{\ell_0+1} & A_M^{\ell_0+2}-A_M^{\ell_0+1} & A_M^{\ell_0+3}-A_M^{\ell_0+1} & \cdots & A_M^{\ell_1}-A_M^{\ell_0+1} & \cdots & A_M^{\ell_{L-1}} & \cdots & A_M^{\ell_L}-A_0^{\ell_{L-1}+1} & = & b_M \\
 & & & 1 & 0 & 0 & \cdots & 0 & & & & & = & 1 \\
 & & & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & & & 1 & \cdots & 0 & = & 1
\end{array}
$$

3.2 Theorem: The number of sets containing two or more basic variables is at most M.

The sets containing two or more basic variables plus S_0 are called essential sets. Given a feasible basis,*

*Obtaining a first feasible solution is accomplished with a phase I procedure as in the usual simplex method.

we assume we have selected for each S_ℓ one basic variable x_{k_ℓ} to be the key variable for that set. We then consider the modified system (3.3) obtained by subtracting, for each set, the key column from every other column in its set (in (3.3) assuming for simplicity that the key variable was the first column in each set). In this modified system, the value of the key variables must be one. We then subtract the coefficients of the key variables from the right hand side as we would variables at upper bound in an upper-bounded variables algorithm [L.P.E., Ch. 18] for the simplex method.

We then introduce the following notation

if $\quad A^j \epsilon S_\ell \quad$ we let

$$D^{k_\ell} = A^{k_\ell}$$

(3.4) $\quad D^j = A^j - A^{k_\ell} \qquad j \neq k_\ell$

and

$$d = b - \sum_{\ell=1}^{L} D^{k_\ell} = b - \sum A^{k_\ell}.$$

Moreover, the last L components of $\underline{b} - \sum \underline{A}^{k_\ell}$ are all zero.

Since the D^{k_ℓ} are incorporated in d, they no longer appear explicitly in our system. The working basis, B, is given by $B = \{D^j / A^j$ is basic and not key$\}$. By virtue of theorem 3.1, it is clear that B has exactly M columns. We let $B^1 = A^0$ so that the first column of B corresponds to the coefficient of the variable to be optimized. We define the derived system to be

$$(3.5) \qquad \sum_{A^j \text{ not key}} y_j D^j = d$$

and it is now easy to prove:

3.6 Theorem: B is a basis for (3.5).

Proof: Suppose $\sum \lambda_j B^j = 0, \quad \lambda \neq 0.$

Since B^j differs from $\underline{B}^j$ only by zero components we have $\sum \lambda_j \underline{B}^j = 0$. But this implies that the $\underline{B}^j$ plus the key columns are linearly dependent since the $\underline{B}^j$ by themselves are. On the other hand, this set is obtained from a (linearly independent) basis by subtraction of columns from within

the set which does not reduce the rank, yielding a contradiction.///

By theorem 3.2 there exist at most M sets with more than one basic variable. These sets and S_0 are the only sets which contain members of B, i.e., they are the <u>essential</u> sets. Thus, with each feasible basis for the original system, we have associated a set of L key variables and a basis for the derived system (3.5). We now show that we can carry out the steps of the simplex method using just the inverse, B^{-1}, of B, the reduced basis, and the corresponding basic solution, $y = B^{-1}d$, of (3.5).

The first step is to obtain a set of prices for the original problem. Let us denote by $\pi = (\pi_1, \ldots, \pi_M)$ the prices on the first M equations and $\mu = (\mu_1, \ldots, \mu_L)$ the prices on the last L. These prices are determined uniquely by the condition that

$$(\pi,\mu)\,\underline{A}^0 = (\pi,\mu)\,\underline{A}^{j_1} = 1$$

$$(\pi,\mu)\,\underline{A}^{j_i} = 0 \qquad i = 2,\ldots,M+L.$$

$\tilde{\pi} = (B^{-1})_1$ has the property that $\tilde{\pi} B^1 = \tilde{\pi} A^0 = 1$, and $\tilde{\pi} B^j = 0 \quad j = 2,\ldots,M$, i.e., $\tilde{\pi}$ is a set of prices for (3.5). To extend $\tilde{\pi}$ to a set of prices for our original problem, we merely set

$$\text{(3.7)} \qquad \tilde{\mu}_\ell = -\tilde{\pi} A^{k_\ell} \qquad \ell = 1,\ldots,L.$$

Because

$$(\tilde{\pi},\tilde{\mu})\ \underline{A}^{j_i} = (\tilde{\pi},\tilde{\mu})\ \underline{A}^{*\ell} = 0 \quad \text{if } A^{j_i} \text{ is key by (3.5) or}$$

$$(\tilde{\pi},\tilde{\mu})\ \underline{A}^{j_i} = (\tilde{\pi},\tilde{\mu})\ (\underline{B}^i + \underline{A}^{*}) \text{ for some } A^{*} \text{ key, } A^{j_i} \text{ not key}$$

$$= (\tilde{\pi},\tilde{\mu})\ \underline{B}^i + (\tilde{\pi},\tilde{\mu})\ A^{*}$$

$$= 0 + 0.$$

Thus $(\tilde{\pi},\tilde{\mu})$ does form a set of prices for the original system.

Using these prices we can "price out" the columns of the original system to find one to enter into the basis. Using the usual simplex criterion, the incoming column A^s would be chosen by

$$\Delta_s = \min_j \Delta_j$$

where

$$\Delta_j = (\pi,\mu)\underline{A}^j = \sum \pi_i A_i^j + \mu_\ell \quad \text{for} \quad A^j \epsilon S_\ell$$

If $\Delta_s \geq 0$, we have an optimal basic feasible solution and we are done; otherwise we bring $\underline{A}^s$ into the basis (assume $A^s \epsilon S_\sigma$). To do this, we must express $\underline{A}^s$ and $\underline{b}$ in terms of the current basis for the original system. If we let

$$\bar{D}^s = B^{-1}D^s = B^{-1}(A^s - A^{*\sigma})$$

then

$$\text{(3.8)} \qquad (A^s - A^{*\sigma}) = \sum_{i=1}^{M} \bar{D}_i^s B^i = \sum \bar{D}_i^s\ (A^{\eta_i} - A^{\nu_i})$$

where η_i indicates the column number in (3.3) corresponding to the i^{th} column of the working basis and ν_i denotes the

column number of the corresponding key variable.

We denote the representation of $\underline{A}^s$ in terms of the current basis by $\underline{\bar{A}}^s$, that is

$\underline{\bar{A}}^s = \sum_{i=1}^{M+L} \bar{A}_i^s \, A^{j_i}$. From (3.8) we see that

$$
\text{(3.9)} \qquad \bar{A}_i^s = \begin{cases} 1 - \sum\limits_{\nu_t = k_\sigma} \bar{D}_t^s & \text{if } A^{j_i} = A^{k_\sigma} \\[2ex] \bar{D}_t^s \ \text{ if } A^{j_i} = A^{\eta_t} & \text{for some } t \\[2ex] - \sum\limits_{\nu_t = j_i} \bar{D}_t^s & \text{if } A^{j_i} = A^{\nu_{t_0}} \text{ for some } \nu_{t_0} \neq k_\sigma \\ 0 \text{ otherwise.} & \end{cases}
$$

The current values for the variables in the basis, $\bar{b}_i$, are found in a similar way. That is, let $\bar{d} = \bar{d}_1, \ldots, \bar{d}_M$ be given by

$$
\text{(3.10)} \qquad \bar{d} = B^{-1} \left(b - \sum A^{k_\ell}\right) = B^{-1} d
$$

then

$$
\left(b - \sum A^{k_\ell}\right) = \sum \bar{d}_i B^i = \sum \bar{d}_i \left(A^{\eta_i} - A^{\nu_i}\right)
$$

and as in (3.9) the $\bar{b}_i$ are given by

$$
\text{(3.11)} \qquad \bar{b}_i = \begin{cases} 1 - \sum\limits_{\nu_t = j_i} \bar{d}_t & \text{if } A^{j_i} \text{ is key} \\[2ex] \bar{d}_t \ \text{ if } A^{j_i} = A^{\eta_t} & \text{for some } t \\[2ex] 0 \ \text{ otherwise.} & \end{cases}
$$

Finding the variable to leave the basis is accomplished in exactly the same way as in the ordinary simplex method. Let

$$\text{(3.12)} \qquad \theta = \frac{\bar{b}_r}{\bar{A}_r^s} = \min_{\bar{A}_i^s > 0} \frac{\bar{b}_i}{\bar{A}_i^s}$$

where we of course require that $\bar{A}_r^s > 0$. Let us assume that $A^{j_r} \epsilon S_\rho$. Then three cases can occur in the updating process:

(a) If $A^{j_r} \epsilon S_\sigma$, the outgoing variable, is the key variable in S_σ, then B remains unchanged, and A^s simply replaces A^{j_r} as the key variable in S_σ. This requires the updating of $\bar{d}$ which is accomplished as follows:

$$\bar{d} := B^{-1}\left(b - \sum_{\ell \neq \sigma} A^{k_\ell} - A^{k_\sigma} + A^{j_r} - A^s\right)^*$$

---

*Where the symbol ":=" does not indicate equality but rather that the expression on the right replaces (updates) the variable on the left.

$$\text{(3.13)} \qquad \begin{aligned} &= B^{-1}\left(\underline{b} - \sum_{\ell = \sigma} A^{k_\ell} - A^{k_\sigma}\right) - B^{-1}\left(A^s - A^{j_r}\right) \\ &= \bar{d} - B^{-1}\left(A^s - A^{j_r}\right) \\ &= \bar{d} - \bar{D}^s \end{aligned}$$

Observing that $A^{j_r} = A^{k_\sigma}$, this is easy to compute since we already have $\bar{d}$, and $\bar{D}^s$ was already generated to determine the $\bar{A}_i^s$.

(b) If A^{j_r} is not a key variable, then we update B^{-1} simply by pivoting on the column $\overline{D}^s$ on the row which $A^{j_r} - A^{*\rho}$ occupies in the inverse of the working basis. In symbols $B^{-1} := PB^{-1}$ where P is the matrix which performs the pivot, $\overline{d}$ is updated by applying P to the old $\overline{d}$.

(c) If $\rho \neq \sigma$ and A^{j_r} is key, we must first change the key variable in S_ρ from A^{j_r}, this changes the working basis and its inverse. To do this, we consider all columns of B corresponding to the set S_ρ, there must be at least one such, since after A^s enters the basis S_ρ must contain a basic variable (Theorem 3.1). One of these columns, call it A^*, is to become the key variable. To obtain the new working basis, $\overline{B}$, from the old one, B, we multiply the column $A^* - A^{j_r}$ in B by -1 to obtain $A^{j_r} - A^*$ and then subtract $A^* - A^{j_r}$ from every other column of the form $A^j - A^{j_r}$ for $j = *$ to obtain $A^j - A^*$. That is, $B := BT$ where T is of the form

$$(3.14) \qquad T = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ 0 \ldots 0 & & -1 \ldots & -1 & -1 & -1 \ldots & -1 & 0 \ldots 0 & & & \\ & & & & & 1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 1 & & & \end{bmatrix}$$

where the -1's occur in the columns corresponding to S_ρ. $B^{-1} := T^{-1}B^{-1}$ and it is easily verified that $T^{-1} = T$ since applying the process twice replaces A^{j_r} as the key variable.

The values for $\bar{d}$ are updated by applying T^{-1}. Now with the new key variable in S_ρ we simply go through step (b). With our updated B^{-1}, $\bar{d}$ and key variables we are now ready to make another iteration.

When -1's appear in the last M + L equations, the algorithm can be changed in a quite obvious way. Theorems 3.1, 3.2, and 3.6 are still valid and we can require that each key variable have a +1 in its last L components since clearly each set must have such a column which is basic. In the pricing process, if A^j is to be priced and has a negative one in one of the last L equations, the appropriate μ is subtracted rather than added to πA^j. To form the difference columns, D^j, the key column is added rather than subtracted from columns with a -1, and appropriate modifications in (3.9) and (3.11) must reflect this. Otherwise the algorithm remains unchanged.

Chapter II. A Geometric Theory of Optimization

1. The fundamental problem: Most of the material in this chapter is based on the following problem.

$$(1.1) \qquad \text{Max.} \quad Z \qquad \text{s.t.} \quad U_0 Z + P = 0$$

where U_0 is the unit vector $(1,0,0,\ldots,0)$, P may be chosen arbitrarily from $\mathcal{K}$, a closed convex set in $m+1$ euclidean space, and Z is a scalar variable. This problem, by the convexity of $\mathcal{K}$, is trivially equivalent to

$$(1.2) \qquad \text{Max.} \quad Z$$

$$\text{s.t.} \quad U_0 Z + \sum_{j=1}^{n} \lambda_j P^j = 0$$

$$\sum \lambda_j = 1$$

$$(\lambda_j \geq 0)$$

$$P^j \in \mathcal{K} \qquad j = 1,\ldots,n$$

where n can be taken arbitrarily large. Formally, we can take the dual of (1.2) obtaining:

$$(1.3) \qquad \text{Min.} \quad \mu$$

$$\text{s.t.} \quad \pi U_0 = \pi_0 = 1$$

$$\pi P + \mu \geq 0 \quad \text{for all } P \in \mathcal{K}.$$

We want to establish duality relations between (1.1) and (1.3) modeled on the results of theorems I.1.5 and I.1.6 in linear programming. In particular: if feasible solutions exist to both (1.1) and (1.3) then min μ = max Z; if feasible solutions exist for one problem and not the other, then the solutions are unbounded in the direction of extremization; and if an optimal solution exists to one, an optimal solution exists for the other and min μ = max Z. Unfortunately all this cannot be accomplished, and the principal obstruction is the problem of the non-zero multiplier, which raises havoc throughout the theory of optimization. It appears at a crucial juncture in proving the "equivalence" of matrix games and linear programming; as we shall see later, it is a motivation for constraint qualifications in non-linear programming; it appears in the question of normality in the calculus of variations; and of course in the theory of optimal control. To make clear what is involved, we consider a very simple geometric interpretation of (1.1), (1.2) and (1.3).

If we let $\mathcal{L} = \{(P_0, P_1, \ldots, P_m)/P_1 = P_2 = \ldots = P_m = 0$, and P_0 arbitrary$\}$ then problem (1.1) may be interpreted as finding the "lowest" point $(-Z, 0, \ldots, 0)$ common to $\mathcal{L}$ and the convex set $\mathcal{X}$ (Figure 1.4).

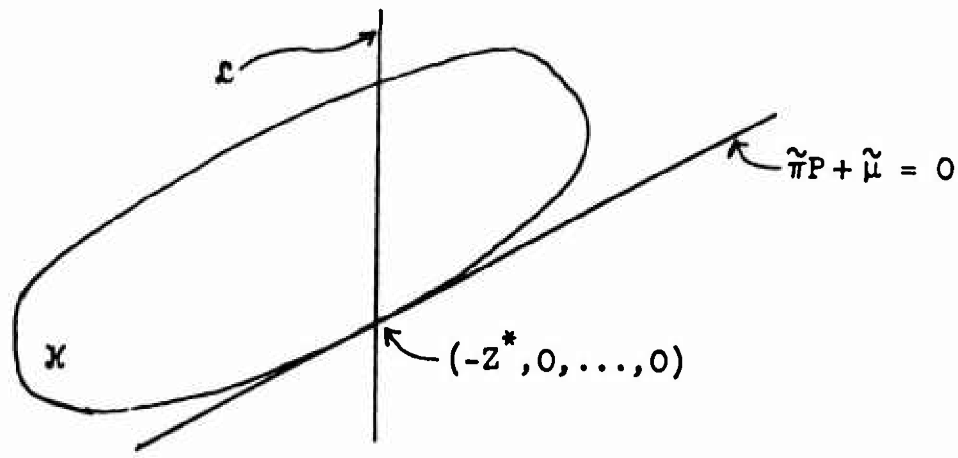
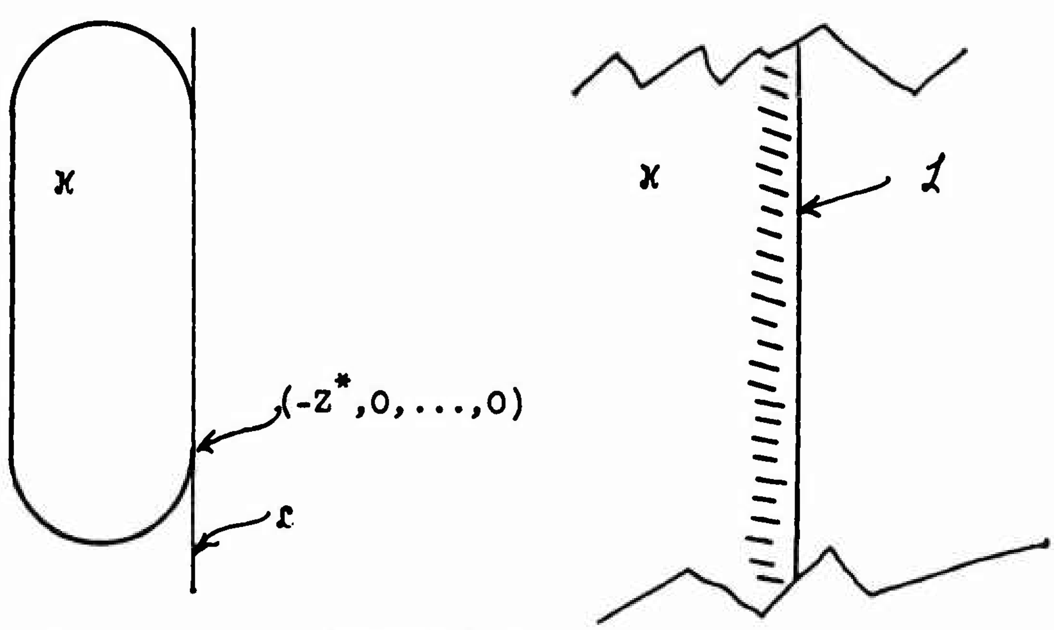


Figure 1.4

Since $\mathcal{K}$ is closed, if such a "lowest" point exists it will be a boundary point of $\mathcal{K}$ and hence belongs to a supporting hyperplane of $\mathcal{K}$, determined, say, by $\tilde{\pi}P + \tilde{\mu} = 0$. Since the representation of the supporting hyperplane is homogeneous in the variables $\tilde{\pi}$ and $\tilde{\mu}$ we can multiply $\tilde{\pi}$ and $\tilde{\mu}$ by a common non-zero multiplier and get an equivalent supporting hyperplane. By the outrageously prejudiced choice of notation in (1.3) and Figure 1.4, it is clear that we wish to convert $\tilde{\pi}$ and $\tilde{\mu}$ into a solution of (1.3), and generally we can. Unfortunately, Figure 1.4 has prejudiced the issue in another way, the "hooker" comes in situations as pictured in Figure 1.5.



(a) Figure 1.5 (b)

In order to satisfy the constraint $\pi U_0 = \pi_0 = 1$ in 1.3 by $\tilde{\pi}$, we must perform the obvious normalization $\pi = \frac{1}{\tilde{\pi}_0} \tilde{\pi}$, $\mu = \frac{1}{\tilde{\pi}_0} \tilde{\mu}$, but in 1.5 (a) the only supporting hyperplane has $\tilde{\pi}_0 = 0$. On the other hand, Figure (b) illustrates a case where there is a supporting hyperplane, but none such that $\tilde{\pi}_0 \neq 0$, and Z is unbounded above (-Z unbounded below). In the "nice" case, Figure 1.4, it is easy to see we can interprete the dual problem (1.3) as the problem of finding the supporting hyperplane "underneath" $\mathcal{X}$ with the highest intercept with $\mathcal{L}$. By "underneath" we mean that for all $P \in \mathcal{X}$, $\pi P + \mu \geq 0$, and the highest intercept is μ. The reason we insist on the "non-zero multiplier" π_0 is that while it is necessary that each optimal solution of (1.1) belongs to a supporting hyperplane, it is also sufficient for a feasible solution to (1.1) to be optimal if it belongs to a "non-vertical" supporting hyperplane, i.e., one in which $\pi_0 > 0$. In cases such as 1.5 (a,b) we cannot resolve the question by using the dual. In particular, in Figure 1.5 (a) the only supporting hyperplane for the optimal feasible solution is vertical, but there are also other feasible solutions belonging to the same supporting hyperplane. So the hyperplane does not uniquely determine the optimal feasible solution, as it does in cases where $\pi_0 > 0$.

If the set $\mathcal{X}$ is polyhedral, as in linear programming, life is easier. As indicated schematically in Figure 1.6,

if there is a vertical supporting hyperplane (π^2,μ^2) at the optimum for (1.1), there is also a non-vertical one, (π^1,μ^1) and, as a matter of fact, a whole manifold of supporting hyperplanes.

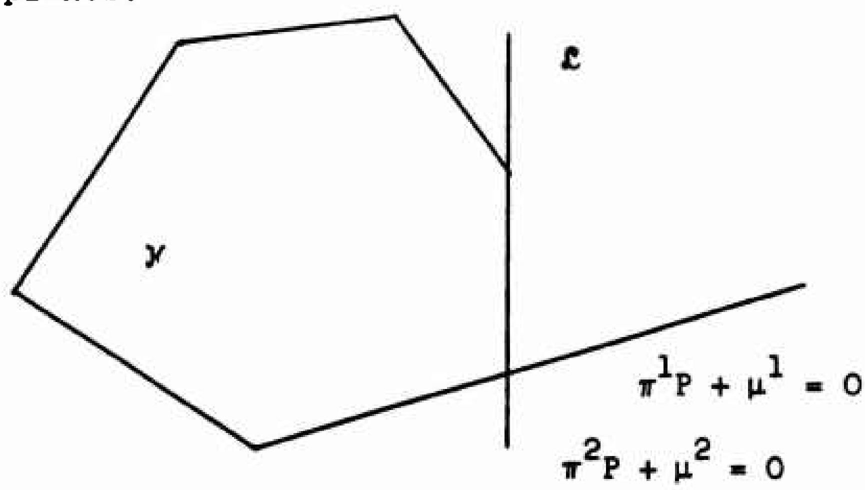


Figure 1.6

This is a result of the "pointedness" of convex polyhedral sets (A.2.13). We turn now from this somewhat heuristic motivation to a more rigorous development of these ideas.

2. A duality theorem for the fundamental problem: Before we prove duality theorems for (1.1) and (1.3) we make some definitions.

2.1 Definition: A set $\mathcal{K} \subset E^{m+1}$ is said to be bounded below if there exists π, μ such that $\pi \epsilon E^{m+1}$, $\pi_0 = 1$ and $\pi P + \mu \geq 0$, for all $P \epsilon \mathcal{K}$.

2.2 Remark: If we interpret the plus x_0 direction as "up", $\mathcal{K}$ being "bounded below" simply means there is a non-vertical hyperplane ($\pi_0 = 1$) which is "below" all points $\mathcal{K}$, i.e., if

(2.3) $P^0 = (P_0^0, \ldots, P_m^0)$ satisfies $\pi P^0 + \mu = 0$ and

if $P^1 = (P_0^1, P_1^0, P_2^0, \ldots, P_m^0) \epsilon \mathcal{K}$, then $P_1^0 \geq P_0^0$.

Requiring $\mathcal{K}$ to be bounded below in (1.1) is then equivalent to requiring that a feasible solution to the dual (1.3) exists.

2.4 Definition: We say that a closed convex set, $\mathcal{K} \subset E^{m+1}$ is regular if the line $\mathcal{L} = \{(P_0, \ldots, P_m)/P_1 = P_2 = \ldots = P_m = 0\}$ intersects the relative interior of $\mathcal{K}$.*

*See Appendix A for material on affine geometry and the dimension of convex sets. In [15a] and [17a] the results of this chapter have been considerably generalized.

2.5 Remarks: Intuitively it is clear that if $\mathcal{L}$ intersects the relative interior of $\mathcal{K}$, there exists a non-vertical supporting hyperplane at the lowest point of $\mathcal{L}$ in $\mathcal{K}$,

Unfortunately, the property of being regular is not invariant under affine transformations. In this sense, being regular is not a characteristic of K, but of K together with the line $\mathcal{L}$. In the case of polygonal sets, regularity is not needed as indicated in section 1 (A.2.13).

To prove our duality theorem, we first need some general results from the theory of convex sets.

2.6 Theorem (Bounding hyperplane): Let K be a convex set belonging to a linear space $\mathcal{S} \subset E^{m+1}$, and let P^0 be any point belonging to the boundary of K, relative to $\mathcal{S}$, then there exists a non-trivial linear homogeneous functional, f, on $\mathcal{S}$ and scalar α such that $f(P) + \alpha \geq 0$, $P \in K$ and $f(P^0) + \alpha = 0$.

Proof: See (A.2.18).

2.7 Corollary: Let K be a closed convex cone with vertex at the origin belonging to a linear subspace $\mathcal{S} \subset E^{m+1}$ and let $P^0 \neq 0$ be any boundary point of K. Then there exists a non-trivial linear functional, f, on $\mathcal{S}$ such that $f(P) \geq 0$, for all $P \in K$, and $f(P^0) = 0$.

Proof: Since $0 \in K$, if we apply 2.6 to K considered as a convex set, we have f and α such that $f(P) + \alpha \geq 0$ for all $P \in K$. $f(0) = 0$ implies $\alpha \geq 0$. Suppose $\alpha > 0$. By Theorem 2.6 $f(P^0) = -\alpha < 0$ and by linearity and the fact that K is a cone, $\lambda P^0 \in K$ and $f(\lambda P^0) = \lambda f(P^0) = -\lambda\alpha$ for all $\lambda \geq 0$. For $\lambda > 1$ $f(\lambda P^0) = -\lambda\alpha < -\alpha$ contradiction.///

2.8 Corollary: If P^0 belongs to the relative boundary of K in Theorem 2.6, there exists f, α satisfying

$f(P^0) + \alpha = 0$, $f(P) + \alpha \geq 0$ for all $P \epsilon K$, such that $f(P) + \alpha > 0$ for all P in the relative interior of K. The same relations hold for Theorem 2.7 with $\alpha = 0$.

Proof: Let $\mathcal{J} = \{Q = P - P_0 / P \epsilon V_k\}$ where V_k is the linear variety of minimal dimension containing K.* $\mathcal{J}$ is then a vector

---

*See Appendix A for discussion of linear varieties of minimal dimension, affine dimension (a.d.), and other material on affine geometry.

---

space of dimension k a.d.(K) with 0 as a boundary point of the set

$$K^0 = K - \{P^0\} = \{Q = P - P^0 / P \epsilon K\}$$

Applying Theorem 2.6 to K^0 in $\mathcal{J}$, we obtain $\tilde{f}$ satisfying $\tilde{f}(Q) \geq 0$, $Q \epsilon K^0$ where $\tilde{f}$ is a linear (homogeneous) functional defined on $\mathcal{J}$ and not identically zero there. Furthermore, $\tilde{f}$ cannot be 0 everywhere on K^0 for if it were, $K^0 \subset \mathcal{R} = \{Q/\tilde{f}(Q) = 0\}$, and $\mathcal{R} \cap \mathcal{J}$ since $\tilde{f} \neq 0$ on $\mathcal{J}$, is a linear space of dimension $< k$ which contradicts the definition of V. Let $Q' \epsilon K^0$ be any point such that $\tilde{f}(Q') > 0$. Let Q be an arbitrary point in the relative interior of K^0 and suppose $\tilde{f}(Q) = 0$. Since Q is in the relative interior of K^0, $Q - \lambda(Q' - Q)$ is in the relative interior of K^0 for some $\lambda > 0$ sufficiently small. But by linearity $0 \leq \tilde{f}(Q - \lambda(Q' - Q)) = (1 - \lambda)\tilde{f}(Q) - \lambda\tilde{f}(Q') < 0$ which is a contradiction. Hence $\tilde{f}(Q) > 0$ for all $Q \epsilon$ relative interior of K^0. We now extend the domain of $\tilde{f}$ to all of E^m. This can easily be done. Let

$\mathcal{W}$ be the orthogonal complement of $\mathcal{S}$. Then $E^m = \mathcal{S} \oplus \mathcal{W} = \{S + W / S\epsilon\mathcal{S},\ W\epsilon\mathcal{W}\}$. For any $P\epsilon E^m$ we define $\tilde{f}(P) = \tilde{f}(S)$ where $P = S + W$ for some $S\epsilon\mathcal{S}$, $W\epsilon\mathcal{W}$. Now we define f by $f(P) = \tilde{f}(P) - \tilde{f}(P^0)$. Then for any $P\epsilon\mathcal{K}$, $\tilde{f}(P - P^0) \geqq 0$ which implies that

$f(P) = f(P - P^0) + f(P^0) = \tilde{f}(P - P^0) - \tilde{f}(P^0) + \tilde{f}(P^0) - \tilde{f}(P^0) = \tilde{f}(P - P^0) - \tilde{f}(P^0) \geqq -\tilde{f}(P^0)$. Strict inequality holding for P in the relative interior of $\mathcal{K}$. Letting $\alpha = \tilde{f}(P^0)$ we are done.///

Finally we come to

2.9 <u>Theorem</u> (Duality): <u>If in (1.1) and (1.3) $\mathcal{K}$ is bounded below and is regular, (1.1) and (1.3) both have optimal solutions and Min μ = Max z.</u>

<u>Proof</u>: Consider $\mathcal{C} = \mathcal{L} \cap \mathcal{K}$. By the hypothesis of regularity, $\mathcal{C}$ is non-empty, and by virtue of $\mathcal{K}$ being bounded below, and $\mathcal{K}, \mathcal{L}$ closed there exists a P^* on $\mathcal{C}$ with the smallest initial co-ordinate. Clearly P^* is on the boundary of $\mathcal{K}$ so by the bounding hyperplane theorem 2.5 there exists $\tilde{\pi}$, $\tilde{\alpha}$ such that

(2.10) $\tilde{\pi}P^* = \tilde{\alpha}$, and $\tilde{\pi}P = \tilde{\alpha}$ for all $P\epsilon\mathcal{K}$

Next we must show that there exists π, α satisfying (2.10) and the additional condition $\pi_0 = 1$. Let $\mathcal{V}$ be the linear variety of lowest dimension containing $\mathcal{K}$ and let k be its dimension. Two cases can occur, depending on whether P^* belongs to the relative interior of $\mathcal{K}$ or not.

Case I: If P^* belongs to the relative interior of K, it follows, because P^* is a boundary point of K, that $k < m+1$ and that $\mathcal{L} \not\subset \mathcal{V}$. There is a 1-1 correspondence between linear m-manifolds in E^{m+1} and relations $\pi y + \alpha = 0 (\pi \neq 0)$. Let $\mathcal{U} = \{x / \pi' x + \alpha' = 0\}$ be any m-manifold containing $\mathcal{V}$ but not $\mathcal{L}$, then $\pi_0' \neq 0$. Let $\pi = \frac{1}{\pi_0'} \pi'$, $\alpha = \frac{1}{\pi_0'} \alpha'$.

Case II: If P^* does not belong to the relative interior of K, there exists by regularity a relative interior point $P^0 \varepsilon C$ and by 2.8 a $\tilde{\pi}$ such that $\tilde{\pi}(P^0 - P^*) > 0$. But for any $P \varepsilon C$, $P_1 = \ldots = P_m = 0$ hence $\tilde{\pi}(P^0 - P^*) = \tilde{\pi}_0 (P_0^0 - P_0^*) > 0$ implying that $\tilde{\pi}_0 > 0$. Again we let $\pi = \frac{1}{\tilde{\pi}_0} \tilde{\pi}$, and $\alpha = \frac{-\tilde{\pi} P^*}{\tilde{\pi}_0}$.

This establishes a feasible solution to 1.3 by taking $\mu = \alpha$. All that remains is to show that this solution is optimal. Multiplying the two equations of (1.1) by π and μ respectively and adding yields

$$\pi U_0 Z + \pi P = 0 \quad \text{or}$$

$$Z + (\pi P + \mu) = \mu \quad \text{and since}$$

$\pi P + \mu \geq 0$ we have $Z \leq \mu$ for any feasible solutions to (1.1) and (1.3). Since the particular values of z and μ for the solutions we have constructed satisfy $z = \mu$ both are optimal.///

2.11 Remark: The bounded-below assumption simply implies the existence of a feasible solution to the dual; the regularity condition among other things implies the existence of a feasible solution to the primal. Dropping these assumptions we now attempt to imitate the unbounded solutions theorem of linear programming (I.1.6).

2.12 Theorem: For $\mathcal{X}$ closed and convex, the following may occur:

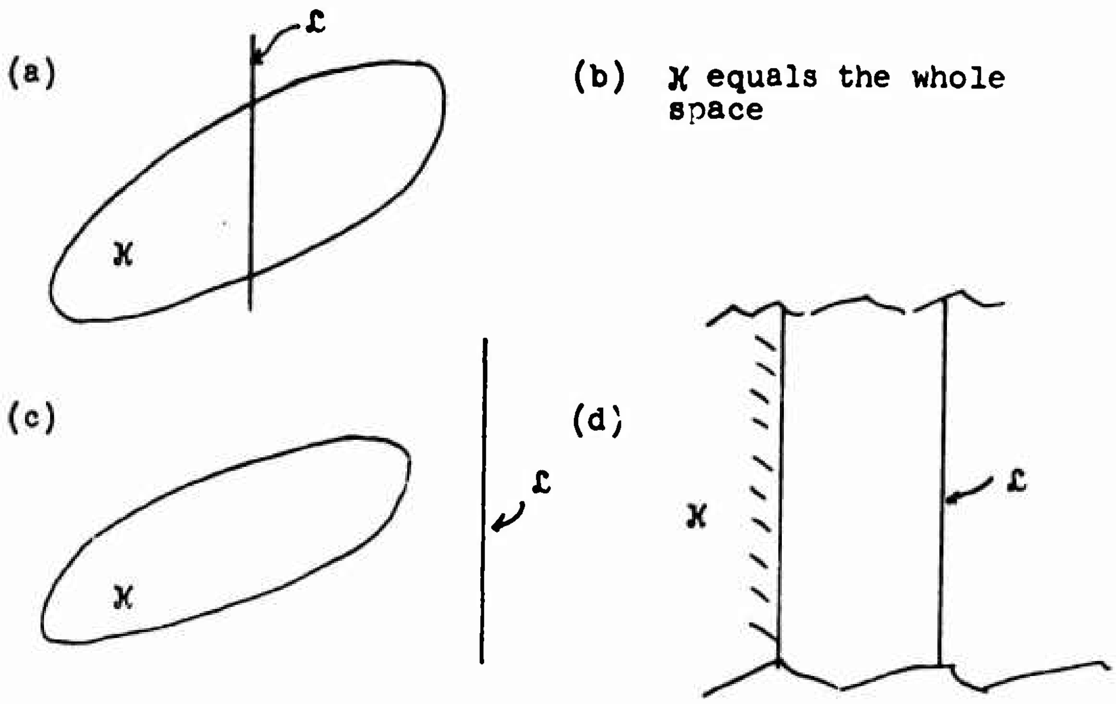
(a) (1.1) and (1.3) both admit feasible solutions.

(b) (1.1) admits feasible solutions but (1.3) does not.

(c) (1.3) admits feasible solutions but (1.1) does not.

(d) Neither (1.3) nor (1.1) admits feasible solutions.

Proof:



2.13 Theorem: If $\mathcal{X}$ is closed and convex exactly one of the following occurs*:

(a) (1.1) and (1.3) are both feasible and Max Z = Inf μ. Further, if $\mathcal{X}$ is regular, Inf μ = Min μ = Max Z.

(b) (1.1) is feasible and (1.3) is not and Sup Z = $+\infty$.

(c) (1.3) is feasible and (1.1) is not, and Inf μ = $-\infty$.

(d) Neither (1.1) nor (1.3) is feasible.

*This theorem is generalized in [17a] .

<u>Proof</u>: (a) If K is regular, the result follows from Theorem 2.9. Suppose K is not regular. As in the proof of 2.9, there is an optimal solution to (1.1), call it P^*. Let $\varepsilon > 0$ be arbitrary, and let $\tilde{P} = (P_0^* - \varepsilon, 0, \ldots, 0)$. $\tilde{P} \notin K$, hence by (A.2.17) there exists $\tilde{\pi}, \tilde{\alpha}$ such that $\tilde{\pi}\tilde{P} + \tilde{\alpha} < 0$, and $\tilde{\pi}P + \tilde{\alpha} \geq 0$ for all $P \in K$, in particular $\tilde{\pi}P^* + \tilde{\alpha} \geq 0$. But $\tilde{\pi}\tilde{P} = \tilde{\pi}_0\tilde{P}_0 = \tilde{\pi}_0(P_0^* - \varepsilon) < \tilde{\pi}P^* = \tilde{\pi}_0 P_0^*$. This implies that $\tilde{\pi}_0 > 0$; we assume without loss of generality that $\tilde{\pi}_0 = 1$. Identifying $\tilde{\alpha}$ with μ in (1.3) we see that $\tilde{\pi}\tilde{P} = \tilde{\pi}_0\tilde{P}_0 = \tilde{P}_0 = P_0^* - \varepsilon < -\alpha = -\mu = P_0^* = \tilde{\pi}P^*$. Since ε is arbitrary $\operatorname{Inf} \mu = -P_0^* = \operatorname{Max} Z$.

(b) If Sup $Z < +\infty$, there exists $\tilde{P} = (\tilde{P}_0, 0, \ldots, 0) \notin K$ with $\tilde{P}_0 < P_0$ for some point $P \in \mathcal{L} \cap K$. Applying (A.2.17) as above we obtain $\tilde{\pi}, \tilde{\alpha}$ such that $\tilde{\pi}P + \tilde{\alpha} \geq 0$, $\tilde{\pi}\tilde{P} + \tilde{\alpha} < 0$. $\tilde{\pi}_0$ can be taken as 1 for the same reasons as above and $\pi = \tilde{\pi}$, $\mu = \tilde{\alpha}$ is a feasible solution to (1.3) contradicting the assumption that no such solutions exist.

(c) Suppose $\tilde{\mu} = \operatorname{Inf}\{\mu \mid$ there exists π such that π, μ satisfies (1.3)$\}$ is finite and let $\tilde{P} = \{-\mu, 0, \ldots, 0\}$. Let $\tilde{K} = \{P = (P_0, \ldots, P_m) \,/\, P_i = P_i', \ i = 1, \ldots, m, \ P_0 \geq P_0'$ for $P' \in K\}$. $\tilde{K}$ is closed and $\tilde{P} \notin \tilde{K}$ since (1.1) is infeasible. Hence there exists $\tilde{\pi}$ such that $\tilde{\pi}\tilde{P} < \operatorname{Inf}\{\tilde{\pi}P / P \in \tilde{K}\}$ (A.2.17). We have, by the definition of $\tilde{K}$, that $\tilde{\pi}_0 \geq 0$ (assuming K non-trivial, if K is empty the result is obvious). If $\tilde{\pi}_0 > 0$, (π, μ) satisfying $\pi = \frac{1}{\tilde{\pi}_0}\tilde{\pi}$, $\tilde{\mu} > \mu > \operatorname{Inf}\{\tilde{\pi}P / P \in \tilde{K}\}$ is a solution of (1.3) with $\mu < \tilde{\mu}$ contradicting the definition of $\tilde{\mu}$.

Now suppose $\tilde{\pi}_0 = 0$. We then have $\tilde{\pi}\tilde{P} = 0$ hence $\tilde{\pi} P > \delta > 0$ for all $P \in K$. Let $\bar{\pi}, \bar{\mu}$ be any feasible solution of (1.3). Then we have $(\bar{\pi} + \lambda\tilde{\pi})P = -\tilde{\mu} + \lambda\delta$. Thus $(\bar{\pi} + \lambda\tilde{\pi}), \bar{\mu} - \lambda\delta$ is dual feasible. But $\bar{\mu} - \lambda\delta$ can be made arbitrarily negative by increasing λ.///

3. The simplex method and the fundamental problem

The algorithm we present here is basically a specialization of the generalized programming procedure outlined in Chapter I. On the other hand we will, in fact, use the results of this section to give a convergence proof for the generalized programming procedure. We consider the fundamental problem in the equivalent form (3.1) where for the time being we take $d^j = 1$.

$$(3.1)\qquad \begin{aligned} &\text{Max} && Z \\ &\text{s.t.} && U_0 Z + \sum_{j=1}^{n} \lambda_j P^j = 0 \\ &&& \sum \lambda_j d^j = 1 \\ &&& \lambda_j \geq 0 \end{aligned}$$

$P^j \epsilon \mathcal{K} \subset E^{m+1}$ and P^j may be chosen arbitrarily from $\mathcal{K}$.

Where we assume that $\mathcal{K}$ is closed, bounded below, and that the problem is regular, we also assume, initially that $\mathcal{K}$ has interior points.

Let $\mathcal{A}$ be the characteristic cone for $\mathcal{K}$(A.2.15), and let $\mathcal{K}^1$ and $\mathcal{A}^1$ be sets with the following properties:

$$\mathcal{K} = \mathcal{A} \oplus \{P/P = \sum \lambda_j P^j, \lambda_j \geq 0, \ \sum \lambda_j = 1, P^j \epsilon \mathcal{K}^1\}$$

$$\mathcal{A} = \{P/P = \sum \lambda_j P^j, \lambda_j \geq 0, \ P^j \epsilon \mathcal{A}^1\}$$

Further, assume without loss of generality that, A^1 is compact.

If we can take $\mathcal{K}^1$ to be compact, we say the $\mathcal{K}$ is <u>compact enough</u>.

We initiate the algorithm by assuming we have $m+1$ fixed points $P^1,\ldots,P^{m+1} \in \mathcal{K}$ such that $\{U_0, P^1,\ldots,P^{m+1}\}$ is a non-degenerate basis for (3.1) where we take $d_1 = d_2 = \ldots = d_{m+1} = 1$. A basis is <u>non-degenerate</u> if every basic variable is strictly positive in the corresponding basic solution. Such a basis must exist by the regularity hypothesis and the assumption of interior points for $\mathcal{K}$. At the end of this section we show how to obtain this initial basis so that $P^1,\ldots,P^{m+1}$ are also extreme points and also what to do if $\mathcal{K}$ has no interior.

The algorithm goes as follows. At iteration k we have $n = m+k$ fixed points P^j. (3.1) is solved, for the fixed choices of P^j, by the simplex method obtaining a point

$$(3.2.a) \qquad Q^k = \sum_{j=1}^{m+k} \lambda_j P^j = (-Z^k, 0, \ldots, 0)$$

as an optimal solution, and dual variables (π^k, μ^k) satisfying

$$(3.2.b) \qquad \text{Min} \quad \mu^k$$

$$\text{s.t.} \quad \pi^k P^j + d^j \mu^k \geq 0 \quad j = 1,\ldots,m+k$$

$$\pi^k U_0 = \pi_0^k = 1.$$

(3.1) is called the <u>master problem</u> and (3.1) with $m+k$ particular choices of P^j the <u>restricted master</u>. We then try to solve the <u>subproblem</u>

$$(3.3) \qquad \underset{P \in \mathcal{K}}{\text{Min}} \quad \pi^k P + \mu^k$$

If Min $(\pi^k P + \mu^k) = 0$, then Q^k given by (3.2.a) is an optimal solution for the problem (3.1) with arbitrary choices for the P^j's rather than just the particular ones in the restricted master. Since the minimum in (3.3) may not exist, we have to be a little careful how we perform the "pricing out". First we observe that if $\pi^k Q < 0$ for $Q \in \mathcal{A}$, and $P \in \mathcal{X}$ that $\pi^k(P + \lambda Q)$ goes to $-\infty$ as λ increases and $P + \lambda Q \in \mathcal{X}$ for all $\lambda \geq 0$. The next observation we make is that

$\inf_{Q \in \mathcal{A}^1} \pi Q < 0$ if and only if $\inf_{Q \in \mathcal{A}} \pi Q < 0$. Clearly

$\inf_{Q \in \mathcal{A}^1} \pi Q \geq \inf_{Q \in \mathcal{A}} \pi Q$. Suppose $Q = \sum_{j=1}^{n} \alpha_j Q^j$ where Q is an arbitrary member of $\mathcal{A}$ and $\alpha_j > 0$ $Q^j \in \mathcal{A}^1$, $j = 1,\ldots,n$.

Then $\pi Q = \sum \alpha_j \pi Q^j$ and clearly $\min_j \pi Q^j < 0$ if $\pi Q < 0$.

Hence there is a Q^j such that $\pi Q^j < 0$. So the first

step is to solve the alternate subproblem

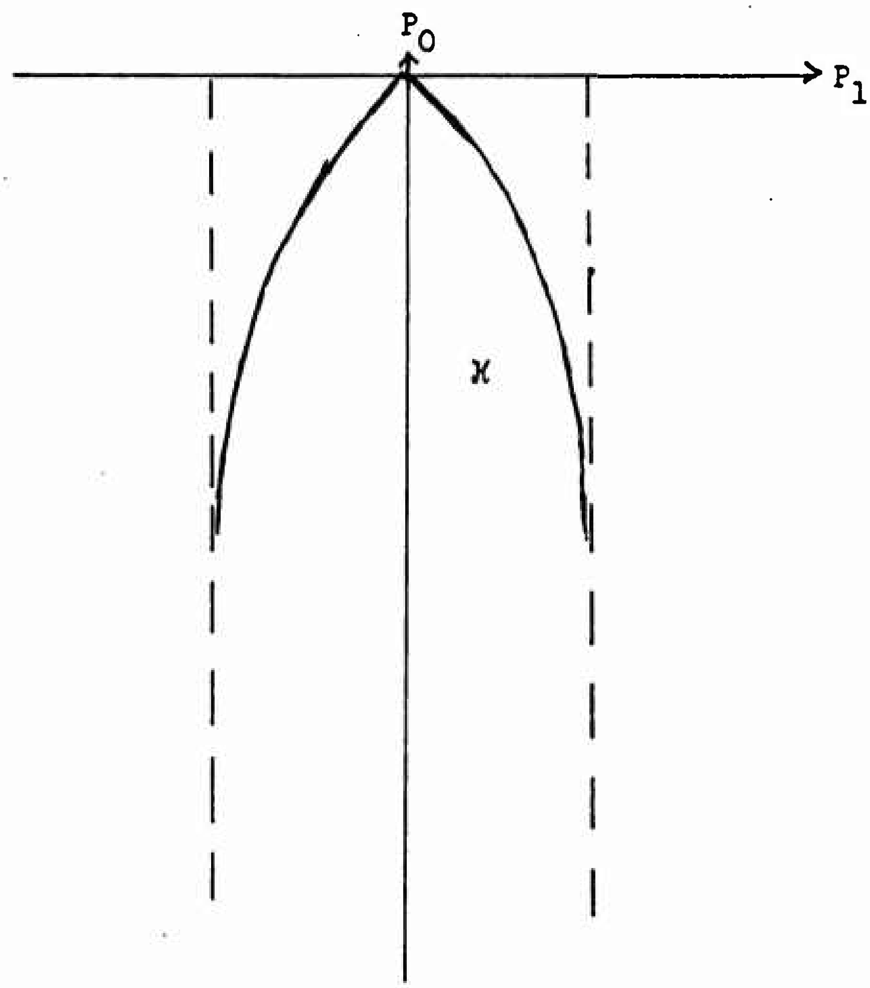
(3.4) $\min_{P \in \mathcal{A}^1} \pi^k P$. Since $\mathcal{A}^1$ is compact the minimum is achieved;

if the minimum is less than zero, we augment (3.1) by an additional column P^{m+k+1} with $d_{m+k+1} = 0$, where P^{m+k+1} is a minimizing vector solution to 3.4. Such a column is called a <u>homogeneous column</u>.

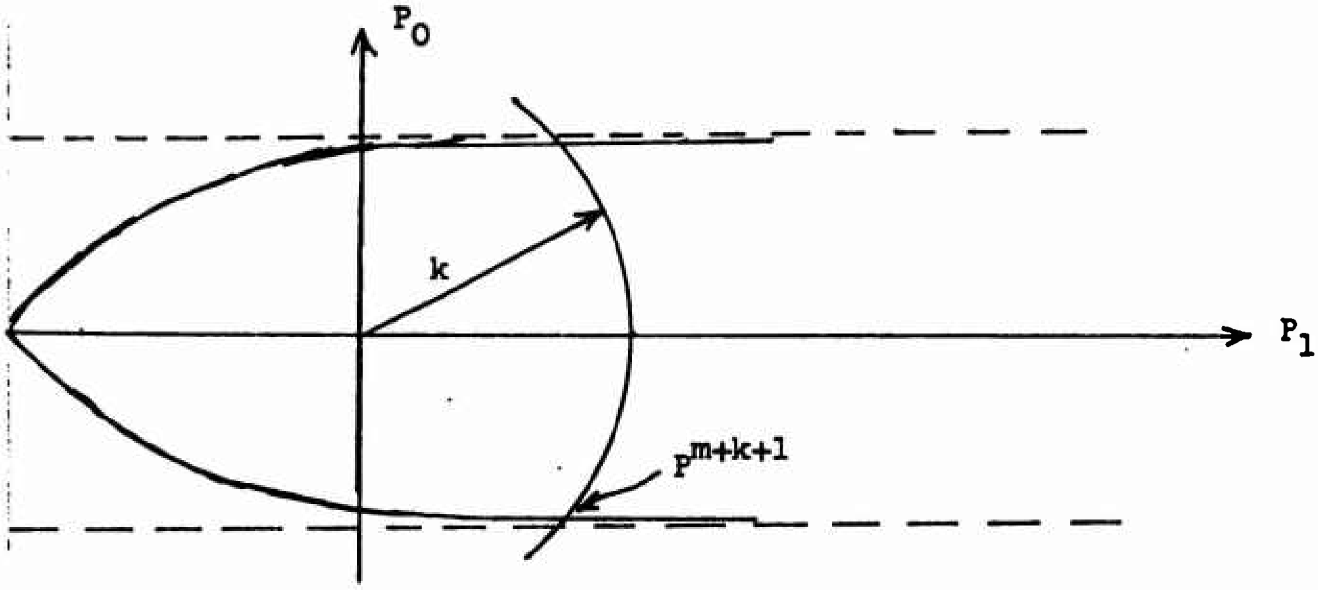
We <u>then</u> consider (3.3), but instead of minimizing over $\mathcal{X}$, we minimize over $\mathcal{X}^1 \cap S(k)$ where $S(k)$ is the sphere of radius k centered at the origin. We use $\mathcal{X}^1$, because by

the same reasoning as for $\mathcal{A}^1$, Inf $\{\pi P / P \epsilon \mathcal{K}\}$ = Inf $\{\pi P / P \epsilon \mathcal{K}'\}$. We bound the set using $S(k)$ since the infimum may not be achieved at a finite point. If $\mathcal{K}$ is compact enough, i.e., $\mathcal{K}^1$ is compact, we need not use the $S(k)$. As an example, all polyhedral sets are compact enough (A.2.7). If Min $\{\pi^k P + \mu^k / P \epsilon \mathcal{K}^1 \cap S(k)\} < 0$ we introduce a minimizing vector P^{m+k+1} with $d_{m+k+1} = 1$. If the minimum is 0, we increase the radius of the sphere by one, and try again. If $\mathcal{K}$ is not compact enough, it is necessary to be able to tell whether Min $\{\pi^k P + \mu^k / P \epsilon \mathcal{K}^1\}$ is equal to zero or not, in order to be able to tell when to terminate the algorithm.

To illustrate the situations which may arise, we consider the following examples. In each case $\pi = (\pi_0, \pi_1) = (1,0)$.



In this example $\mathcal{A} = \{(x_0,x_1)/x_0 = -\lambda, x_1 = 0, \lambda \geq 0\}$, thus if $\mathcal{A}^1 = \{(-1,0)\}$ we would introduce $(-1,0)$ as the next (homogeneous) column with $d_{m+k+1} = 0$. In the second example, $\mathcal{A} = \{(0,\lambda)/\lambda \geq 0\}$ so $\text{Min}\,\{\pi Q / Q \in \mathcal{A}\} = 0$.



In (3.3) $\text{Inf}\,\{\pi P / P \in \mathcal{K}\} = -1$ which however is not attained. Thus we truncate at a distance k from the origin and introduce P^{m+k+1} with $d^{m+k+1} = 1$.

We note that if $\mathcal{K}$ is a cone, then $\mathcal{K} = A \oplus \{P\}$ where P is the vertex of the cone and at every iteration we consider only (3.4). If on the other hand, $\mathcal{K}$ is compact, we need consider only (3.3) without truncation.

We will show in Theorem 3.7 that this process converges on a subsequence to optimal solutions of both (1.1) and 1.3) whenever $\mathcal{K}$ is bounded below and regular.

3.5 Lemma: If $\mathcal{K}$ is bounded below and regular, there is an accumulation point (π,μ) of the sequence (π^k,μ^k).

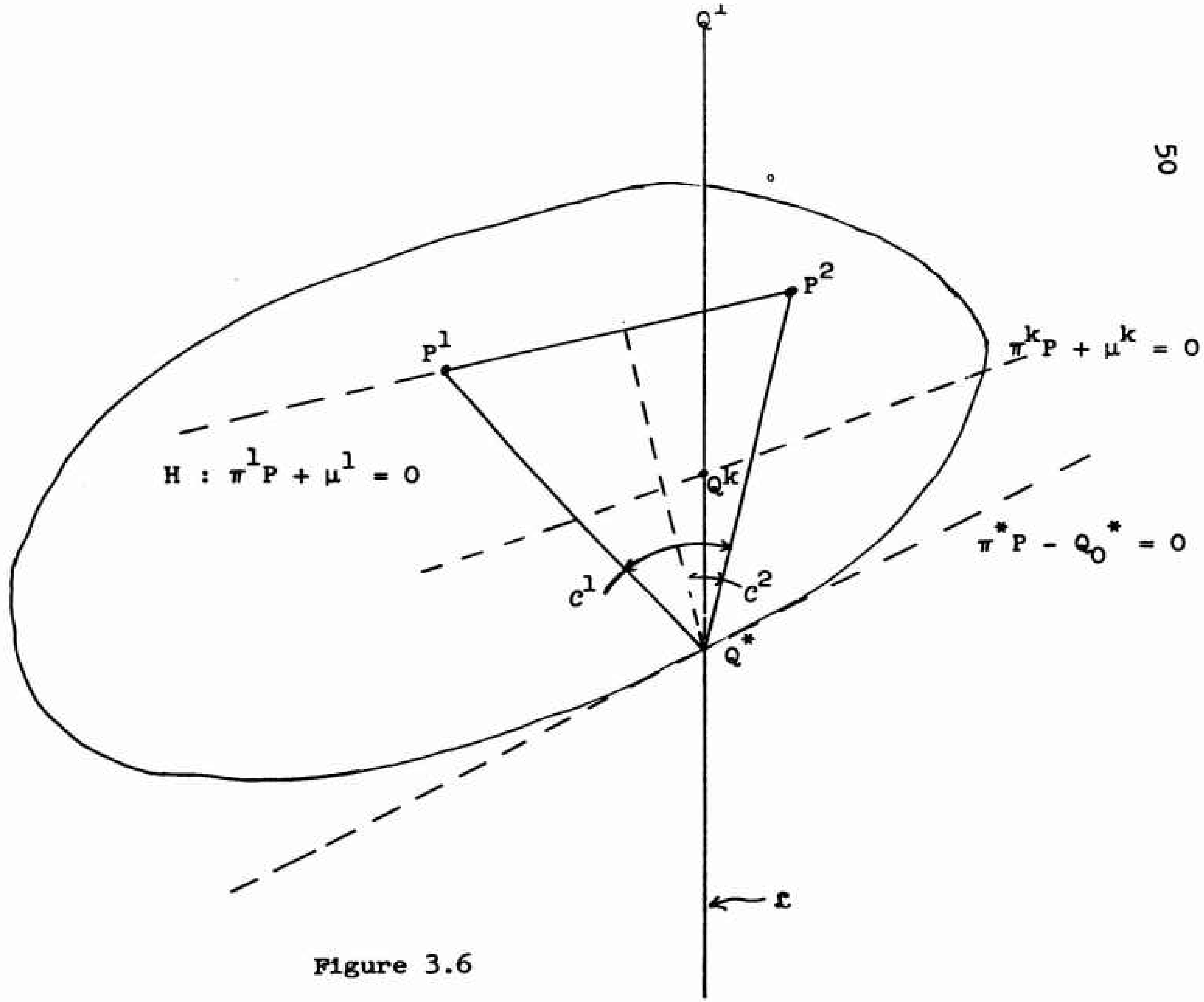


Figure 3.6

Proof: Referring to (3.2.a) and (3.2.b) we note that $-\mu^k \geq -\mu^{k+1} \geq Q_0^*$, for $k = 1,2,\ldots$, where Q^* is the optimal solution of (1.1). Hence there is a limit point μ of the sequence μ^k.

Consider the plane, H, defined by $\pi^1 P + \mu^1 = 0$. By the non-degeneracy hypothesis, Q^1 given by (3.2.b), see Fig. 3.6, is a relative interior point of the m-dimensional simplex in H determined by $P^1,\ldots,P^{m+1}$. Hence the ray $Q^* + \lambda(Q^1 - Q^*)$ for $\lambda > 0$ is an interior ray of the cone, $\mathcal{C}^1$, eminating from Q^* which is "generated" by linear combinations of the vectors $P^j - Q^*$, $j = 1,\ldots,m+1$. Hence there is a "circular" cone, $\mathcal{C}^2$, of the form $\mathcal{C}^2 = \{P + Q^* / P_1^2 + P_2^2 + \ldots + P_m^2 \leq \epsilon^2 P_0^2\}$ for some $\epsilon > 0$, contained in $\mathcal{C}^1$. Any dual variables π^k, μ^k must satisfy $\pi^k P + \mu^k \geq 0$ for any P^j, $j = 1,\ldots,m+k$ and thus any P of the form

$$P = \sum_{j=1}^{m+1} \lambda_j P^j, \ \lambda_j \geq 0 \quad \sum \lambda_j = 1. \quad \text{Since } \mu^k \leq -Q_0^*$$

$\pi^k P - Q_0^* \geq 0$ for these P. But this hyperplane goes through Q^* and hence $\pi^k P - Q_0^* \geq 0$ for all $P \in \mathcal{C}^2$. If we let $P = (1,\pm\epsilon,0,0,\ldots,0) + Q^*$ which belongs to $\mathcal{C}^2$, then

$$\pi^k P - Q_0^* = \pi^k(1,\epsilon,0,\ldots,0) + \pi^k Q^* - Q_0^*$$

$$= \pi_0^k \pm \epsilon\pi_1^k = 1 \pm \epsilon\pi_1^k \geq 0$$

which implies that $|\pi_1^k| \leq \frac{1}{\epsilon}$, applying the same argument

to $|\pi_2^k|, \ldots, |\pi_m^k|$ we show that each component of π^k is bounded in absolute value, hence there is a convergent subsequence of π^k.///

The situation is depicted schematically in Fig. 3.6.

3.7 Theorem: There exists a convergent subsequence of the π^k, μ^{k*} such that the limit π, μ and $(-\mu, 0, \ldots, 0)$ satisfy (1.3) and (1.1) respectively.

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*By a common abuse of notation, we denote this subsequence also by π^k, μ^k.

Proof: By lemma 3.5 there is a convergent subsequence with limit π, μ. We will show that (π, μ) satisfies (1.3) and hence by the duality theorem 2.9 $(-\mu, 0, \ldots, 0) = Q^*$ satisfies (1.1) and $Q^k \to Q^*$, where Q^k is the optimal solution for the problem (3.1) at the k^{th} iteration. Let us consider the subsequence which corresponds to the iterations which result in homogeneous columns being introduced for the next iteration. We assert that

$\pi P \geq 0$ for all $P \in A$ or equivalently A^1. Suppose not, then $\pi P^* < -\delta$ for some $P^* \in A^1, \delta > 0$. Since $\pi^k \to \pi$,

$$\min_{P \in A^1} \pi^k P = \pi^k P^{m+k+1} < -\delta/2, \text{ for all } k > k_0$$

for some k_0. On the other hand, $\pi^\ell P^{m+k+1} \geq 0$ for all $\ell > k$ since π^ℓ is a dual solution for the linear programming problem at the ℓ^{th} iteration. Subtracting, we obtain

$$(\pi^\ell - \pi^k) P^{m+k+1} > \delta/2$$

Letting ℓ and k go to infinity (such that $\ell > k$) on the convergent subsequence, we have a contradiction since $\lim (\pi^{\ell} - \ell^{k}) = 0$.

$k < \ell$

$k \to \infty$

Similarly we show that

$\pi P + \mu \geq 0$ for all $P \varepsilon K^1$. Suppose the contrary were true. Then there exists $P^* \varepsilon K^1$ such that $\pi P^* + \mu < -\delta < 0$. Then for some k_0

$$\pi^k P^{m+K+1} + \mu^k = \min_{\substack{P \varepsilon K_1^1 \\ A \varepsilon A^1}} \pi^k P + \mu^k$$

$$< -\delta/2, \text{ for } k \geq k_0.$$

Subtracting and passing to the limit in the same way as before, we obtain a contradiction. Thus since any column of K can be written as a positive sum of elements in A^1 plus a convex sum of elements of K^1 we have $\pi P + \mu \geq 0$ for all $P \varepsilon K$.///

We now turn to the problem of getting a starting solution. The idea is to adjoin to K the artificial points $\tilde{P}^j(M) = (M, \tilde{P}_1{}^j, \ldots, \tilde{P}_m{}^j)$ $j = 1, \ldots, m+1$ where $(\tilde{P}_1^j, \ldots, \tilde{P}_m^j)$ $j = 1, \ldots, m+1$ form a simplex in E^m containing 0_m as an interior point. We denote by $K(M)$ the convex hull of K and the points $\tilde{P}^j(M)$. We then use these points as the starting basis. If M is taken large enough it can easily be seen that the algorithm will converge to an optimal feasible

solution, and after a finite number of steps the artificial points will no longer be in the basis.

If we call the above problem the rectified problem, we have then the following theorem:

3.8 Theorem: For any problem (1.1) which is regular and such that $\mathcal{K}$ has a non-empty interior, there exists an M_0 such that for all $M > M_0$ the generalized simplex algorithm applied to the rectified problem converges to an optimal solution of the original problem if one exists, and after a finite number of iterations the artificial variables can be discarded.

Proof: We will not give all the details but will outline the essential steps. Since $\mathcal{K}$ is regular, bounded below and has a non-empty interior, there are linearly independent points $\overline{P}^1,\ldots,\overline{P}^{m+1}$ and a point Q^1 belonging to the interior of $\mathcal{K} \cap \mathcal{L}$ which is a strictly positive linear combination of $\overline{P}^1,\ldots,\overline{P}^{m+1}$. We can then go through the same construction as in the proof of lemma 3.5 to obtain a right circular cone $\mathcal{C}^2(M) = \{P + Q^*(M)/P_1{}^2 + \ldots + P_m{}^2 \leqq \epsilon^2(M)P_0{}^2\}$, for each M, where $Q^*(M)$ is the optimal solution for the rectified problem. Clearly if $M_1 < M_2$, we have

$$Q_0{}^*(M_1) \leqq Q_0{}^*(M_2) \leqq Q_0{}^*, \text{ and}$$

$$\epsilon(M_1) \leqq \epsilon(M_2) \leqq \epsilon (= \epsilon(\infty)).$$

Let M_1 be any real number, let $E = \max_j \{\overline{P}_0{}^j\}$

$D = \max_j \| (\tilde{P}_1^j, \ldots, \tilde{P}_m^j) \|$, and

$M_0 = \max \left[E, Q_0^*(M_1) + \frac{D}{\epsilon(M_1)}\right]$. Then referring to Fig. 3.9 we see that $\tilde{P}^j(M_0)$ belongs to

$C^2(M_1) = \{P + Q^*(M_1)/P_1^2 + \ldots + P_m^2 \leq \epsilon^2(M_1)P_0\}$

$j = 1,\ldots,m+1$, and as a matter of fact, $\tilde{P}^j(M_0) \epsilon C^2(M)$ for all $M \geq M_0$ including (passing to the limit) $M = \infty$. Clearly any dual variables π,μ which are an optimal solution to the dual of the rectified problem must satisfy $\pi P + \mu \geq 0$ for all $P \epsilon C(\infty) \cap \mathcal{X}(M)$. But for $M > M_0$, $\tilde{P}^j(M)$ belongs to the interior of $C^2(\infty)$ and hence $\pi\tilde{P}^j + \mu > 0$ (2.8) $j = 1,\ldots,m+1$. Hence, by the convergence of π^k,μ^k to some such π,μ, after a finite number of steps k_0, $\pi^k\tilde{P}^j + \mu^k > 0$ $j = 1,\ldots,m+1$, $k > k_0$. But for any basic variable P for the k^{th} iteration, we must have $\pi^k P + \mu^k = 0$.///

The argument is depicted schematically in Fig. 3.9.

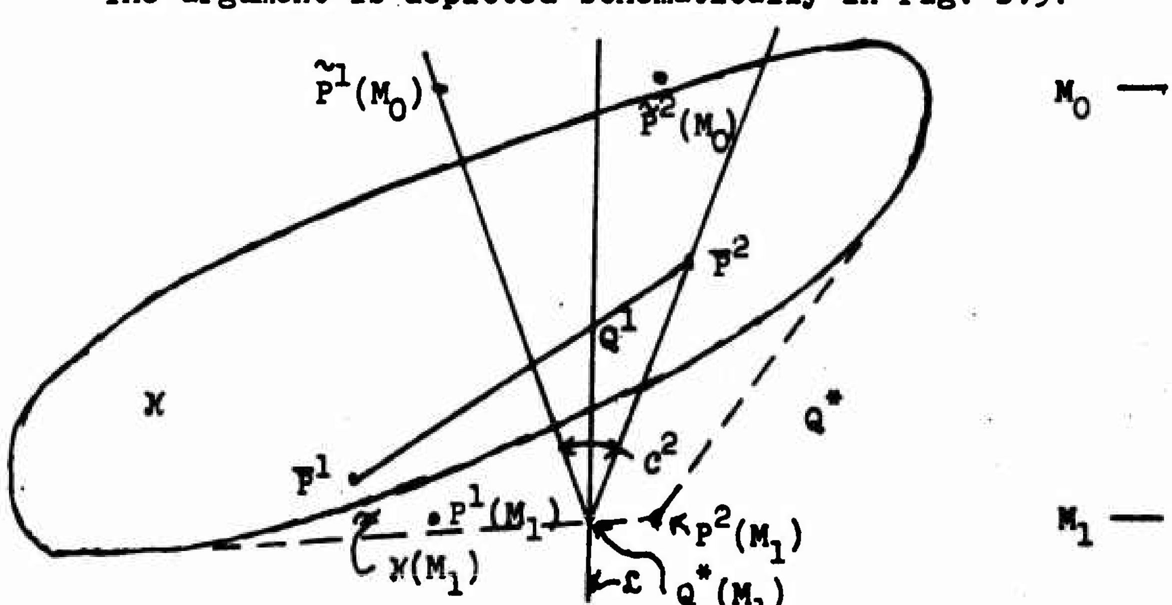


Figure 3.9

This "phase I" procedure however depends on the fact that M_0 is known in some a priori manner. In case $\mathcal{K}$ is not full dimensional, we simply introduce the $m+1$ linearly independent points $\tilde{P}^1,\ldots,\tilde{P}^{m+1}$ as before and if M is large enough the process will converge to an optimal solution to the original problem. However, in this case, a number of $\tilde{P}^j$'s equal to the deficiency of $\mathcal{K}$ will remain in a basis although the value of the corresponding variables will converge to zero.

We now turn to some applications.

4. Applications to linear and convex programming, and to the decomposition principle

The first application we consider is to linear programming. We consider the linear programming problem in the form:

$$(4.1) \qquad \text{Max} \quad Z$$
$$\text{s.t.} \quad U_0 Z + Ax = b$$
$$x \geq 0$$

We let $\mathcal{K} = \{P = Ax - b / x \geq 0\}$ which is a polyhedral cone with vertex at $-b$.

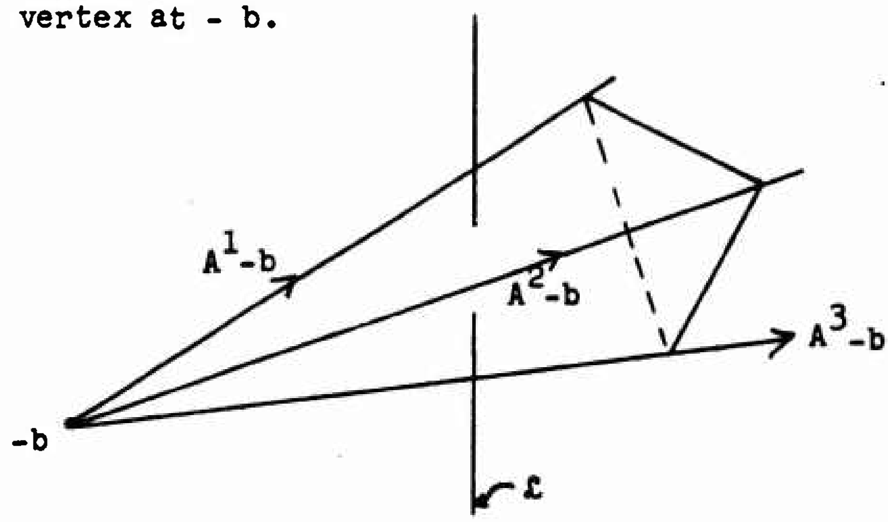


Figure 4.2

Usually in linear programming we take $\mathcal{K}^1 = \{-b\}$, $\mathcal{A}^1 = \{A^1, \ldots, A^n\}$, and moreover take $P^1, \ldots, P^m$ as homogeneous columns and do not use the $\sum \lambda_j = 1$ equation at all since we are dealing with a cone. Since we have one less equation we need one less point in our starting solution. Furthermore

since $K = \left\{ \sum \lambda_j A^j : \lambda_j \geqq 0 \; A^j \varepsilon A^1 \right\}$ is polyhedral we do not have to worry about regularity because of the "pointedness" property (A.2.13) of polyhedral sets. In passing, we may note that the regularity assumption for linear programs of full rank is equivalent to assuming the existence of a non-degenerate basic feasible solution [L.P.E., p.81]. With these conventions the generalized simplex method becomes the ordinary simplex method and the duality theorem 2.9 becomes the corresponding one for linear programming, Theorem I.1.5. To see this we notice that any supporting hyperplane of K determined by π, μ passes through the vertex $-b$ of K. That is $\pi(-b) + \mu = 0$, or $\mu = \pi b$ and at optimality $\min \mu = \pi b = \max Z$.

The choice of A^1 given above is not the only possible choice, a common scaling device is to take

$A^1 = \left\{ \frac{A^1}{||A^1||}, \ldots, \frac{A^n}{||A^n||} \right\}$ which is just the projection

of the columns of A onto the unit sphere. This can be varied by taking various choices for the norm.

A second application is to convex programming, in particular to the convex programming algorithm of G.B. Dantzig [L.P.E., Ch.24]. In fact, it was this algorithm which motivated the approach used in this chapter and, in particular, the proof of the convergence of the generalized

simplex method is a rather straightforward generalization of the convergence proof given by Dantzig for his algorithm.

The problem we consider here is the following:

Find $x = (x_1, \ldots, x_n) \epsilon \mathcal{R}$ a compact enough convex set in E^n and Minimum w satisfying

$$
(4.3) \qquad
\begin{aligned}
L_i(x) &= 0 \qquad i = 1,2, \ldots, r \\
\phi_i(x) &\leq 0 \qquad i = r+1, \ldots, m \\
\phi_0(x) &= w \text{ (Min)}
\end{aligned}
$$

where $\phi_i(x) = L_i(x)$ for $i = 1, \ldots, r$ are linear (inhomogeneous) and $\phi_i(x)$ for $i = 0$, and $i = r+1, \ldots, m$ are continuous convex functions (not necessarily differentiable).

Let $K = \{(P_0, \ldots, P_m)/P_i = \phi_i(x) = L_i(x) \quad i = 1, \ldots, r,$ $P_i \geq \phi_i(x) \quad i = 0, r+1, \ldots, m,\ x \epsilon \mathcal{R}\}$. Clearly K is convex and closed; but K need not be regular or bounded below. Dantzig assumes that $\mathcal{R}$ is bounded which guarantees that K is bounded below and makes the

4.4 Non-degeneracy assumption: The homogeneous parts of the L_i are linearly independent and there exists a point $x^0 \epsilon \mathcal{R}$ such that $L_i(x^0) = 0$, $i = 1, \ldots, r$ and $\phi_i(x^0) < 0$ for $i = r+1, \ldots, m$.

4.5 Theorem: The non-degeneracy assumption 4.4 implies that (4.3) is regular.

Proof: Since $P \epsilon K$ implies that $P_1, \ldots, P_r = 0$, K is at most $m+1-r$ dimensional. Let $P^0 = (\phi_0(x^0), 0, \ldots, 0, \phi_{r+1}(x^0), \ldots, \phi_m(x^0))$. Then by definition of K if P is such that

$P_0 \geq P_0^0$, $P_1 = \ldots = P_r = 0$, $P_{r+1} \geq P_{r+1}^0$, $\ldots$, $P_m^0 \geq P_m^0$ then $P \epsilon K$. The set of all such P is obviously a cone of dimension $m+1-r$ containing the point $P^1 = (\phi_0(x_0)+1, 0, \ldots, 0)$ as an interior point. $P^1 \epsilon \mathcal{L}$/// The following conditions are equivalent to (4.4): The homogeneous parts of L_1 are linearly independent and there exists points x^1, $i = r+1, \ldots, m$ in K such that $L_j(x^1) = 0$, $j = 1, \ldots, r$ and $\phi_1(x^1) < 0$, $i = r+1, \ldots, m$.

Thus the duality theorem 2.9 applies to the convex programming problem and the generalized simplex method converges to an optimal solution.

The application of our procedure to the decomposition principle [L.P.E., Ch.23] is straight forward. A decomposition problem can be formulated

$$\text{(4.6)} \qquad \begin{aligned} &\text{Max} \quad Z \\ &\text{s.t.} \quad U_0 Z + A_1 x = b_1 \\ &\qquad\qquad\quad A_2 x = b_2 \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}$$

where we assume A_2 is of a very special structure, e.g., of transportation structure or a matrix which can be decomposed so that if $A_2 x = b_2$ were the only constraints involved, the solution would be quite easy. We can also apply the theory if $\{x \mid A_2 x = b_2, x \geq 0\}$ is replaced by an arbitrary convex set. The constraint $U_0 Z + A_1 x = b_1$ on the other hand is arbitrary.

In this case

$K = \{A_1 x - b_1 / A_2 x = b_2, x \geq 0\}$

$K^1 = \{A_1 x - b_1 / x$ is an extreme point for the constraint

set $A_2x = b_2$, $x \geq 0\}$, and $\mathcal{A} = \{A_1x$ / where $A_2x = 0$, $x \geq 0\}$.

With these definitions, the subproblem (3.3) becomes

$$\text{(4.7)} \qquad \begin{aligned} \text{Min} \quad & \pi^k (A_1x - b_1) + \mu^k \\ \text{s.t.} \quad & A_2x = b_2 \\ & x \geq 0 \end{aligned}$$

which is by hypothesis easy to solve. If (4.7) is unbounded below a homogeneous column from $\mathcal{A}$ is found which is introduced. Thus the generalized simplex method applied to (4.6) is simply the decomposition algorithm of Dantzig and Wolfe. For the linear decomposition problem we need not introduce a point of K^1 at each iteration. On the other hand, if $\{x \mid A_2x = b_2, x \geq 0\}$ is replaced by a nonpolyhedral convex set we must introduce points of K^1 sufficiently often to guarantee convergence to an optimal solution.

5. Generalized Programming:

As a generalization of (1.1) consider

$$(5.1)\qquad \begin{aligned} &\text{Max} \quad Z \\ &\text{s.t.} \quad U_0 Z + \sum P^j = 0 \\ &\qquad\qquad P^j \epsilon \mathcal{K}^j \qquad j = 1,\dots,n \end{aligned}$$

where each $\mathcal{K}^j$ is a closed, convex set in E^{m+1}. (5.1) has the natural dual

$$(5.2)\qquad \begin{aligned} &\text{Min} \sum_{j=1}^{n} \mu_j \\ &\text{s.t.} \quad \pi P^j + \mu_j \geq 0 \qquad j = 1,\dots,n \qquad \forall\, P^j \epsilon \mathcal{K}^j \\ &\qquad \pi U_0 = 1 \end{aligned}$$

For these programs we have

5.3 Theorem: If $\mathcal{K} = \sum \mathcal{K}^j = \{P/P = \sum P^j, P^i \epsilon \mathcal{K}^j\}$ is closed, regular and bounded below then (5.1) and (5.2) both have optimal solutions, and $\text{Min} \sum \mu_j = \text{Max } Z$.

Proof: Apply the duality theorem 2.9 to

$$(5.4)\qquad \begin{aligned} &\text{Max} \quad Z \\ &\text{s.t.} \quad U_0 x + P = 0 \\ &\qquad\qquad P \epsilon \mathcal{K} = \sum \mathcal{K}^j. \end{aligned}$$

This gives us π^*, μ^*, and P^* which are optimal solutions to the dual and primal respectively. P^* is clearly an optimal solution to (5.1). Since $P^* \varepsilon K$ there exists a representation in terms of the K^j. Denote one such by

$$P^* = \sum P^{*j} \quad \text{where } \pi^* P^* + \mu^* = 0,\ \pi^* P + \mu^* \geqq 0 \text{ for all } P \varepsilon K$$

Let $\mu_j = -\pi^* P^{*j}$,

then $\pi^* P^j + \mu_j \geqq 0$, for all $P^j \varepsilon K^j$ for each $j = 1, \ldots, n$. If not there would exist some $\bar{P}^{j_0}$ such that $\pi^* \bar{P}^{j_0} + \mu_{j_0} \leqq -\varepsilon < 0$.

Let $P = \sum_{j \neq j_0} P^{*j} + \bar{P}^{j_0}$ then

$$\begin{aligned} \pi^* P + \mu^* &= \sum_{j \neq j_0} \pi^* P^{*j} + \pi^* \bar{P}^{j_0} + \mu^* \\ &\leqq \sum_{j=1}^{n} \pi^* P^{*j} - \varepsilon + \mu^* \\ &= -\varepsilon < 0 \text{ contradiction.} \end{aligned}$$

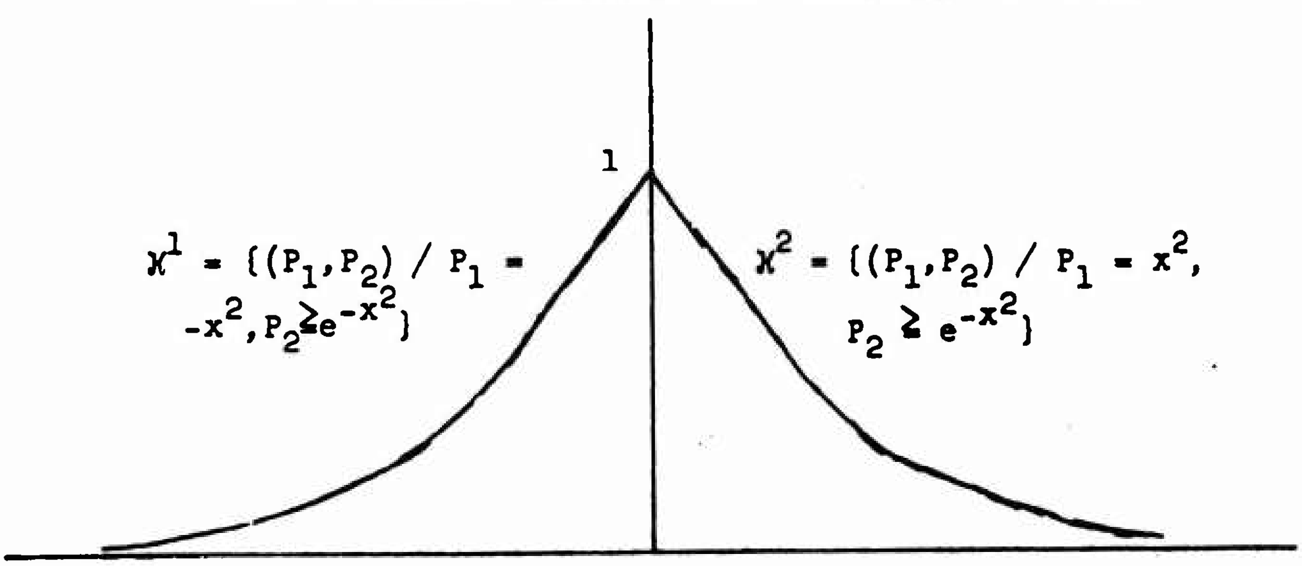
Hence π^*, μ_j^* $j = 1, \ldots, n$ is feasible in (5.2). Conversely, let π, μ_j, $j = 1, \ldots, n$ be any feasible solution to (5.2) and let $\mu = \sum \mu_j$ then π, μ is feasible in (5.4) and hence $\mu \geqq \mu^*$. For $\mu_j = \mu_j^*$ equality holds.///

From the proof of the previous theorem we have immediately:

5.5 Corollary: For every P^*, π^*, μ^* which are optimal solutions to the (5.4) and its dual respectively, there corresponds optimal solutions P^{*j}, π^*, $\mu^*_j = \pi^* P^{*j}$ $j = 1,\ldots,n$ of (5.1) and (5.2) respectively where $P^* = P_1^* + \ldots + P_n^*$. Conversely, if P^{*j}, π^*, μ^*_j $j = 1,\ldots,n$ are optimal solutions for (5.1) and (5.2) respectively, then $P^* = \sum P^{*j}, \pi^*, \sum \mu_j^*$ are optimal solutions to (5.4) and its dual respectively.

Remark: It may turn out that the direct sum of a finite number of closed convex sets is no longer closed and in this case the correspondence in corollary 5.5 may not be valid. If, however, at most one of the $\mathcal{X}^j$ is not bounded, then corollary 5.5 holds.

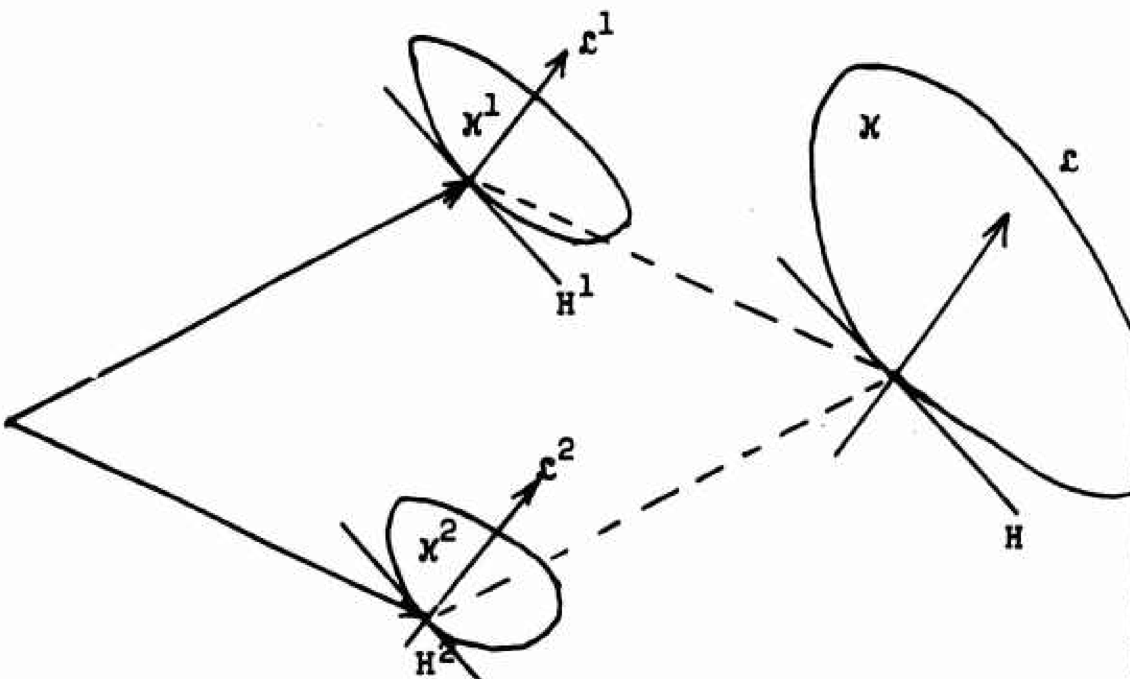
As an example, consider the following two sets:



The origin is a limit point of $\mathcal{X} = \mathcal{X}^1 \oplus \mathcal{X}^2$ but cannot be expressed as the sum of $P^1 + P^2$, $P^1 \epsilon \mathcal{X}$, $P^2 \epsilon \mathcal{X}^2$.

Geometrically we can illustrate theorem 5.5 in the two dimensional case in the following way:

Theorem: Let $\mathcal{K}^1$ and $\mathcal{K}^2$ be two closed convex subsets of the plane with $\mathcal{K}^1$ bounded; and let $\mathcal{K} = \mathcal{K}^1 \oplus \mathcal{K}^2$ be their direct sum. Let $\mathcal{L}$ be any directed line intersecting the relative interior of $\mathcal{K}$ and let P^* be the minimal point on $\mathcal{K} \cap \mathcal{L}$ relative to $\mathcal{L}$. Then there exist parallel supporting lines H, H^1, H^2 to $\mathcal{K}$ at P^*, to $\mathcal{K}^1$ at P^{*1} and to $\mathcal{K}^2$ at P^{*2} respectively such that $P^* = P^{*1} + P^{*2}$ and P^{1*} and P^{2*} are minimal points on $\mathcal{K}^1 \cap \mathcal{L}^1$, $\mathcal{K}^2 \cap \mathcal{L}^2$ respectively where $\mathcal{L}^1$ and $\mathcal{L}^2$ are parallel and in the same direction as $\mathcal{L}$.



Proof: Let $\tilde{\mathcal{K}} = \{P - P^* / P \in \mathcal{K}\}$, $\tilde{\mathcal{K}}^i = \{P - P^{*i} / P \in \mathcal{K}^i\}$ i = 1,2. Apply the identification in Corollary 5.5, to the problem for $\tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}^1$, $\tilde{\mathcal{K}}^2$.///

The algorithm we use for solving (5.1) is the obvious one. We assume we have $P^{j,i}$ $i = 1,\ldots,n+m+1$ which together with U_0 forms a basis for

$$(5.6) \qquad \text{Max} \quad Z$$

$$\text{s.t.} \quad U_0 Z + \sum_{j=1}^{n} \sum_{i=1}^{n_j} \lambda_{j,i} P^{j,i} = 0$$

$$\sum_i \lambda_{j,i} d_{j,i} = 1 \qquad j = 1,\ldots,n$$

$$P^{j,i} \in \mathcal{X}^j, \; d_{j,i} = 0 \text{ or } 1.$$

For the k^{th} iteration we solve (5.6) as a linear program obtaining dual variables π^k, μ^k_j $j = 1,\ldots,n$. We then solve the n-subproblems

$$(5.7) \qquad \Delta_j = \min \; \pi^k P^j + \mu^k_j$$

$$\text{s.t.} \; P^j \in \mathcal{X}^j$$

and let $\Delta = \min_j \Delta_j$. If $\Delta \geq 0$, we are done using Theorem 5.3. If not, suppose

$$0 > \Delta = \Delta_{j_0} = \pi^k P^{j_0} + \mu^k_{j_0}, \quad \text{then } P^{j_0} \text{ is}$$

introduced into (5.6) with the corresponding $d = 1$. If the infimum is not attained, we make the same modifications as in Section 3 either truncating $\mathcal{X}$ or introducing a homogeneous column.

We now indicate a convergence proof for this algorithm. The fact that $\mathcal{X} = \sum \mathcal{X}^j$ is bounded below implies that

each $\mathcal{X}^j$ is bounded below. Hence $\mu^k{}_j$ is bounded below. On the other hand $\sum \mu^k{}_j = \mu^k$ is monotonic non-increasing, so clearly for some $j_0, \mu^k_{j_0}$ is bounded above and converges on a subsequence. Excluding μ_{j_0} from the sum, the same argument may be repeated for the rest of the components. Hence there is a subsequence of the $\mu^k{}_j$ converging to μ_j. There is a convergent subsequence of the π^k converging to some π by the reasoning as in Lemma 3.5. Then applying the same sort of argument used in Theorem 3.7, we can show that π, μ_j form an optimal solution for the dual problem (5.2). If we let $Q^k = \sum_j Q^{j,k}$ be the optimal solution at the k^{th} iteration, we have $Q^k \to P^*$ the optimal solution of 5.4. Further, we know that $\pi Q^k + \sum \mu_j \to \pi P^* + \sum \mu_j = 0$ and $\pi Q^{j,k} + \mu_j \to 0$. This does not imply however that the $Q^{j,k}$ have convergent subsequences. If we have that, each $\mathcal{X}^j$ is compact except for possibly one, then the $Q^{j,k}$ do converge on a subsequence.

Now we consider the generalized linear program of Wolfe and Dantzig which is

$$(5.8) \qquad \text{Max} \quad Z$$

$$\text{s.t.} \quad U_0 Z + \sum x_j Q^j = Q^0$$

$$x_j \geq 0$$

$$Q^j \in \mathcal{O}_j \qquad j = 1,\ldots,n, \text{ where } \mathcal{O}_j \text{ is}$$

a closed convex subset of E^{m+1}.

If we make the identification

$$\mathcal{K}^j = \{\rho Q^j / Q^j \in \mathcal{O}_j,\ \rho \geq 0\} \qquad j = 1,\dots,n$$

and $\mathcal{K}^0 = \{-Q^0\}$, (5.8) is equivalent to

$$\begin{array}{lll} (5.9 & \text{Max} & Z \\ & \text{s.t.} & U_0 Z + \sum \lambda_j P^j - Q^0 = 0 \\ & & \qquad \lambda_j \geq 0 \qquad j = 0,\dots,n \\ & & \qquad P^j \in \mathcal{K}^j \qquad j = 1,\dots,n \end{array}$$

where the restraints $\lambda_j = 1$; $j = 1,\dots,n$ are eliminated since $\mathcal{K}^j$, $j = 1,\dots,n$ are cones. The formal dual of (5.8) is, by applying (5.2) to (5.9) and making the proper identifications:

$$\begin{array}{lll} (5.10) & \text{Min} & \mu = \pi Q^0 \\ & \text{s.t.} & \pi Q^j \geq 0 \qquad Q^j \in \mathcal{O}_j \qquad j = 1,\dots,n \\ & & \pi U_0 = \pi_0 = 1 \end{array}$$

We then prove the following:

5.11 <u>Theorem</u>: <u>If there exists a π, such that $\pi_0 = 1$, $\pi Q^j \geq 0$ for all $Q^j \in \mathcal{O}_j$, $j = 1,\dots,n$ and if the cone generated by $\mathcal{O}$ is closed and the line $Q + tU_0$, intersects its relative interior then both (5.8) and (5.10) have optimal solutions and Min μ = Max Z.</u>

<u>Proof</u>: We simply show that the hypotheses imply 2.9 for the problem (5.9).

$\pi Q^j \geq 0 \Rightarrow \pi \rho P^j \geq 0$, $\rho \geq 0$. If $-\pi Q^0 \geq 0$ set $\mu = 0$ otherwise set $\mu = \pi Q^0$ then

$\mathcal{K} = \left(\sum\right) \mathcal{K}^j \oplus \mathcal{K}^0$ is bounded below. $\mathcal{K}$ is obviously closed and regular.///

5.12 Corollary: If $\mathfrak{A}_j$ is compact $j = 1,\ldots,n$ and if the line $Q^0 + tU_0$, t real, intersects the relative interior of the cone generated by $\mathfrak{A}$, then both (5.8) and (5.10) have optimal solutions and Min μ = Max Z.

Proof: Cones generated by bounded closed sets are closed (A.2.14), and the direct sum of a finite number of closed cones with common vertex is closed. To see this we consider without loss of generality, the case for $n = 2$. Suppose $x^j + y^j \to z$ where $x^j \epsilon \mathcal{K}_1$, $y^j \epsilon \mathcal{K}_2$. Since the unit sphere is compact, there exist x, y and a subsequence of the integer j_i such that $x^{j_i} \to ||x^{j_i}|| \dfrac{x}{||x||}$, $y^{j_i} \to ||y^{j_i}|| \dfrac{y}{||y||}$ and $x^{j_i} + y^{j_i} \to z$, $i = 1,2,\ldots.$ Since $\mathfrak{A}_1$, $\mathfrak{A}_2$ are closed $x \epsilon \mathcal{K}_1$, $y \epsilon \mathcal{K}_2$, and the closed cone spanned by x and y is in $\mathcal{K}_1 + \mathcal{K}_2 = \{\rho_1 P^1 + \rho_2 P^2 / \rho_1, \rho_2 \geq 0,\ P^1 \epsilon \mathfrak{A}_1, P^2 \epsilon \mathfrak{A}_2\}$.///

The generalized simplex method in this case becomes the generalized programming technique of Wolfe and Dantzig [L.P.E., Ch.22] ard converges.

In the next chapter we discuss in detail the application of the generalized simplex method to optimal control theory.

Ch. III The Theory of Optimal Control for Mathematical Programmers

0. Introduction: We can think of a system of ordinary differential equations

(0.1)

$$\frac{dx(t)}{dt} = f(x(t),u(t),t) \qquad x(t) = (x_0(t),\dots,x_n(t))\in E^{n+1}$$

$$u\in\Omega \qquad u(t)\in E^r$$

$$x(0)\in X^0 \qquad t\in[0,\infty)$$

as a dynamical system which allows us to travel from a point in X^0 at time $t = 0$ to various other points at later times. To reach these points we may make choices for the initial point $x(0)$ among those of X^0 and may choose various "control functions" u among those in Ω. For certain special cases of (0.1) the set of points, $S_T(X^0)$, which can be reached at time T is convex. If so, then consider an optimization problem such as minimizing $x_0(T)$ for some fixed T over points belonging to S_T and the line $\mathcal{L} = \{(x_0,\dots,x_n) / x_1 = \dots = x_n = 0\}$ *
In this case

---

* A special case is minimizing

$$\int_0^T f_0(x_1(t),\dots,x_n(t),u(t),t)dt \quad \text{subject to}$$

$$\frac{dx_j(t)}{dt} = f_j(x_1(t),\dots,x_n(t),u,t), \quad x_j(T) = 0 \qquad j = 1,\dots,n.$$

Simply define $x_0(t)$ by $\frac{dx_0(t)}{dt} = f_0(x(t),u(t),t)$, $x_0(0) = 0$.

we have a supporting hyperplane to the set of reachable points at time T including $(x_0^*(T),0,\ldots,0)$, the optimizing point. To find the proper supporting hyperplane the generalized simplex method can be used very successfully especially if the system (0.1) is linear in u and x. Moreover, since the solutions of ordinary differential equations depend continuously on the value of the starting point, it is generally true that an optimal trajectory $x^*(t)$ must be a boundary point of S_t for all t, $0 \leq t \leq T$. Thus if we have the supporting hyperplane $[\Pi(t),\mu(t)]$ for $0 \leq t \leq T$ we know that $\Pi(t)x^*(t) + \mu(t) = 0$. Such a "traveling" hyperplane can be obtained as a solution of the adjoint equation of (0.1) see (B.3). For linear problems, additional hypotheses can be imposed to guarantee that the condition $\Pi(t)x^*(t) + \mu(t) = 0$ determines $x^*(t)$ uniquely [15; Ch.3],[11].

Mention should be made of the degree of generality of the control systems considered for the various theorems. The rule here has been to consider each theorem for as general a system as possible without confusing the basic ideas of the proof involved. Hence the generalized simplex method is applied only to a very simple linear optimal control problem in order not to confuse the basic method with technical details concerned with linearization, generalized trajectories and the like.

In this chapter much use is made of papers in the literature of control theory [7, 10, 11, 12, 14, 15, 16]. Frequently these papers treat the so called time optimal control problem. This problem is the one of reaching a set X^T in E^{n+1} starting at time 0 from X^0 in minimum time. Most of the theory carries over since the essential characteristic of both problems is that the terminal point $x(T)$ belongs to the boundary of the set of reachable points. However, the generalized simplex method does not apply directly and modifications of the method must be introduced. For that reason we will not treat the minimum time problem here.

1. The Structure of $S_T(X^0)$: We assume for all $u \epsilon \Omega$ that $f(x,u,t)$ satisfies the conditions of Theorems B.1.3 and B.1.6 for the existence and uniqueness of solutions through every point (x,t) of $E^{n+1} \times [0,T]$. More particularly, we assume that:

A1. $u \epsilon \Omega$ implies that $u(t)$ is a bounded and measurable function of t;

A2. $f(x,u,t)$ is measurable in t for each fixed x,u and is continuous jointly in x,u for each fixed t. For any bounded region, R, of $E^n \times E^r$ there is an M such that $||f\ (x,u,t)\ || \leq M$ for almost all $t \epsilon [0,T]$ and $(x,u) \epsilon R$.

A3. f satisfies a Lipshitz condition locally in x.

We now formally define the set of reachable points at time T, $S_T(X^0)$ by

(1.1) $S_T(X^0) = \{x^T /$ there exists an absolutely continuous $x(t)$, $0 \leq t \leq T$, such that $x(0) \epsilon X^0$, $\frac{dx}{dt} = f(x,u,t)$ a.e. in t for some $u \epsilon \Omega$ and $x(T) = x^T\}$.

The assumptions A1, A2, and A3 imply only the existence of local solutions. If f changes too rapidly, it may not be possible to define a solution on the entire time interval $[0,T]$ passing through an arbitrary point of $E^n \times [0,T]$. To get around this difficulty and at the same time guarantee that $S_T(X^0)$ is bounded for bounded X^0 we add some additional assumptions. First, we prove the following lemma.

1.2 Lemma: Let $w(t)$ be an absolutely continuous scalar function such that whenever $w(t) > 0$, we have $\frac{dw(t)}{dt} \geq 0$

<u>for all t where the derivative exists. Then if $w(0) > 0$, we have that $w(t) > 0$ for all $t \in [0,T]$.</u>

<u>Proof</u>: $w(t) = w(0) + \int_0^t g(\tau)d\tau$ where $g(\tau)$ is a measurable function of τ and $\frac{dw}{dt} = g(\tau)$ a.e. [9, p.207]. Since $w(0) > 0$, by the continuity of $w(t)$, $w(t) > 0$ on some non-degenerate interval $[0,t_1]$. Let $t^* = \sup\{t / w(\tau) \geq \frac{1}{2}w(0)$ for $\tau \in [0,t]\}$. If $t^* < T$, we have $w(t^*) = w(0) + \int_0^{t^*} g(\tau)d\tau \geq w(0)$. This follows because $w(t) \geq \frac{1}{2}w(0) > 0$ implies that $g(\tau) \geq 0$ a.e. on $[0,t^*]$. This is a contradiction. Hence $t^* = T$.///

We now make some additional assumptions.

A4. Suppose there exists $V(x,t) : E^{n+1} \times [0,T] \to R$ which is positive for all t and $x \neq 0$, and $V(0,t) \geq 0$. Suppose V has continuous first partial derivatives in x and t in $E^{n+1} \times [0,T]$. Furthermore, suppose that $||x|| \to \infty$ implies that $V(x,t) \to +\infty$ uniformly on $[0,T]$. Assume further that there exists $G(V,t)$ which is a strictly increasing function of V for each t such that $\frac{d}{dt} V(x(t),t) \leq G(V,t)$ for all solutions, $x(t)$, of (0.1). Finally, we assume that $\frac{d}{dt} v(t) = G(v(t),t)$ has a solution $v^*(t,v^0)$ on the interval $[0,T]$ for every non-negative initial condition $v^*(0) = v^0 \geqslant 0$.

1.3 <u>Example</u>: An easily applied special case occurs [7] when we have

(1.4) $x \cdot f(x,u,t) \leq C(||x||^2 + 1)$ for all x,t, and $u \in \Omega$. Take $V(x,t) = (||x(t)||^2 + 1)$ and $G(V,t) = 2CV$

Then $\frac{d}{dt} V(x(t),t) = 2\,||x(t)||\,\frac{d}{dt}\,||x(t)||$

$$= 2\,\|\,x(t)\,\|\,\frac{d}{dt}\left(\sum x_i^{\,2}\,(t)\right)^{1/2}$$

$$= 2\,\|\,x(t)\,\|\,\frac{\sum x_i \dot{x}_i}{\|\,x\,\|}$$

$$= 2x \cdot \frac{d}{dt}\,x = 2x \cdot f \leq 2C\,(\|\,x(t)\,\|^2 + 1) = 2CV$$

and $\frac{dv}{dt} = 2Cv$ has the solution $v(t) = v(0)e^{2Ct}$.

1.5 <u>Theorem</u>: <u>If A1 - A4 are satisfied and X^0 is compact, then $S_T(X^0)$ is bounded and for every $u \epsilon \Omega$ and $x^0 \epsilon E^{n+1}$, there exists a unique solution of $\frac{dx}{dt} = f(x,u,t)$ a.e. satisfying $x(0) = x^0$.</u>

<u>Proof</u>: The assumptions A1 - A3 guarantee that any local solution can be extended in a unique way to the boundary of any bounded region in $E^{n+1} \times [0,T]$ (B.1.4). Thus either a local solution $x(t)$ is defined on $[0,T]$ as asserted or there exists t_0 such that $\lim_{t \to t_0} \|\,x(t)\,\| \to \infty$. Let $v^0 = \sup_{x \epsilon X^0} V(x,0)$. For any solution $x(t)$ let

$W(t) = v^*(t;\ v^0 + 1) - V(x(t),\ t)$ for the t for which $x(t)$ is defined.

Then $\frac{d}{dt}\,W \geq G(v^*,t) - G(V,t)$. By the monotonicity of $G(V,t)$ with respect to V, $G(v^*,t) - G(V,t) \geq 0$ if $v^* \geq V$; but $W(0) \geq 1$ hence lemma 1.2 applies and $W(t) \geq 0$ for all t for which $x(t)$ is defined. Thus $v^*(t;v^0+1)$ is an upper bound for $V(x(t),\ t)$ for these same t. But if

$$\lim_{t \to t_0} \|\,x(t)\,\| = \infty \quad \text{then} \quad \lim_{t \to t_0} V(x(t),t) = \infty.$$

But $\lim_{t \to t_0} V(x(t), t) \leq \lim_{t \to t_0} v^*(t; v^0+1)$

$$= v^*(t_0, v^0+1) \text{ which}$$

is finite, which proves the second assertion. The first assertion follows from the fact that $v^*(T; v^0+1)$ is an upper bound to $V(x(T), t)$ for any solution $x(t)$ to (0.1) and the assumption that $\|x\| \to \infty \Rightarrow V(x,T) \to \infty$.///

1.6 Corollary: Let $f(x,u,t) = A(t)x(t) + u(t)$ where each component of $A(t)$ is measurable and bounded. Suppose for all $u \in \Omega$, $\|u(t)\| \leq L$ a.e. and that A1 is satisfied, then $S_T(X^0)$ is bounded if X^0 is compact.

Proof: Let $K = \operatorname*{ess\,sup}_{t\in[0,T]} \| A(t) \|$,

$G(V,t) = 2(K+L)V$ and $V(t) = \| x(t) \|^2 + 1$. Then

$$\frac{d}{dt}V = 2 \| x \| \frac{d}{dt} \| x \| = 2(\sum x_i^2)^{1/2} \frac{d}{dt}(\sum x_i^2)^{1/2}$$

$$= 2\,[x \cdot (Ax + u)]$$

$$\leq 2 \| x \| (K \| x \| + \| u \|) \text{ a.e.}$$

$$\leq 2KV + 2LV \quad \text{since} \quad \| x \|^2 + 1 \geq \| x \|$$

$$= 2(K+L)V \text{ and apply theorem 1.5.///}$$

Remark: 1.6 can be proved directly. Solutions of (0.1) for $f = Ax + u$ are of the form

$x(T) = Y(T)\,x(0) + \int_0^T Y(T)Y^{-1}(s)u(s)ds$ where $Y(T)$ is the matrix solution of $\frac{dY}{dt} = A(t)\,Y$ satisfying $Y(0) = I$ (B.2.5). Since $x(0)$, $Y(T)$, $Y^{-1}(s)$ and $u(s)$ are bounded so is $x(T)$ for $x(0)\in X$, $u\in\Omega$.

We next turn to the question of whether $S_T(X)$ is closed or not. Here we follow Fillippov [7]. It will be necessary to consider the class of admissible "f's", i.e., $G = \{f(x,u,t) / u\in\Omega\}$ as well as Ω itself. We now introduce the assumptions:

A5. $f(x,u,t)$ and $\nabla_x f(x,u,t)$ exist and are continuous in x,u, and t.

A6. $\Omega = \{u / u(t) \in Q(x,t),\ u(t)$ measurable and bounded$\}$ where $Q(x,t)$ is closed and bounded for each x and t and is upper semi-continuous as a function of x and t.

A7. $R(x,t) = \{f(x,u,t) \,/\, u \in \Omega\}$ is convex for each x and t.

Remarks: By the continuity of f in u, $R(x,t)$ is upper semi-continuous in x and t, where by definition, a set valued function $\Phi(s)$ is upper semi-continuous if for any s and any $\epsilon > 0$ there exists a $\delta = \delta(s,\epsilon) > 0$ such that every point of $\Phi(s')$ is within a distance ϵ of some point of $\Phi(s)$ for all $||\, s-s' \,|| < \delta$.

1.7 Theorem: If A1 - A7 are satisfied, $S_T(X^0)$ is closed.

Proof: Let $x^1(T)$, $x^2(T)$, ... be any sequence of points in $S_T(X^0)$, corresponding to trajectories $x^1(t)$, $x^2(t)$, ...
By assumption and the proof of Theorem (1.5)

$$|| \, x^j(t) \, || \leq \sup_{[0,T]} v^*(t,v^0) < \infty, \text{ for } v^0 > \sup_{x \in X^0} V(0,x).$$

On the other hand $\left|\left|\frac{dx^j(t)}{dt}\right|\right| = ||\, f(x^j(t),\, u^j(t),\, t) \,|| \leq M$ almost everywhere for some M. This follows from the upper semi-continuity of $R(x,t)$. For suppose that for $t^j \to t$, $\xi^j \to \xi$ that $||\, f(\xi^j,u^j,t^j) \,|| \to \infty$, then for t^j,ξ^j sufficiently close to t,ξ $R(\xi^j,t^j)$ is contained in an ϵ-neighborhood of $R(\xi,t)$ which is bounded since $Q(\xi,t)$ is bounded and f is continuous; a contradiction. These two facts imply that there exists a subsequence $x^j(t)$ which is equicontinuous, hence we can extract a uniformly convergent subsequence converging to $x(t)$, which is absolutely continuous and such that $\left|\left| \frac{dx}{dt} \right|\right| \leq M$ a.e. These last two facts obtain because $x^j(t)$, $j = 1,2,...$ each satisfy a Lipshitz condition with the same constant M.

Let $\frac{dx(t)}{dt} = y(t)$, and $\frac{dx^j(t)}{dt} = y^j(t) \quad j = 1,2,\ldots.$ Then $y, y^1, \ldots,$ are defined almost everywhere, are measurable and bounded. Let t_0 be a point at which $\frac{dx(t_0)}{dt}$ exists. We then show that $\frac{dx(t_0)}{dt} \in R(x(t_0), t_0)$.

Since $R(x,t)$ is u.s.c. with respect to inclusion, for any $\epsilon > 0$, there is a $\delta > 0$ such that $R(x,t) \in U_\epsilon$, a closed ϵ neighborhood of $R(x(t_0), t_0)$ whenever $|t-t_0| < \delta$ and $|x-x(t_0)| < 2M\delta$. By making δ smaller if necessary, we can assume that

$$(1.8) \quad \left\| \frac{x(t) - x(t_0)}{t-t_0} - \frac{dx(t_0)}{dt} \right\| < \epsilon$$

for $|t-t_0| < \delta$. On the other hand

$$\frac{x(t) - x(t_0)}{t - t_0} = \lim_{i \to \infty} \frac{x^i(t) - x^i(t_0)}{t - t_0}$$

$$= \lim_{i \to \infty} \frac{1}{t-t_0} \int_{t_0}^{t} y^i(\tau)d\tau$$

$$(1.9) \qquad = \lim_{i \to \infty} \int_0^1 y^i(t_0 + (t-t_0)s)ds.$$

For almost all τ.

$$y^i(\tau) = \frac{dx^i(\tau)}{dt} = f(x^i(\tau), u^i(\tau), \tau) \in R(x^i(\tau), \tau).$$

Also for i large enough and t satisfying $|t_0 - t| < \delta$ we have $\| x^i(t_0) - x(t_0) \| < M\delta$ and $\| x^i(t_0) - x^i(t) \| < M\delta$; hence $\| x^i(t) - x(t_0) \| < 2M\delta$ implying that $R(t, x^i(t)) \subset U_\epsilon$. Thus for i large enough and $|t - t_0| < \delta$ the integrand of (1.9) is contained in U_ϵ for almost all s and hence the limit of the integrals belongs to U_ϵ (U_ϵ is convex). Consequently $\frac{x(t) - x(t_0)}{t - t_0} \in U_\epsilon$ and from (1.8) we have that $\frac{dx(t_0)}{dt} \in U_{2\epsilon}$. Since $R(t_0, x(t_0))$ is closed and ϵ is arbitrary

$$\frac{dx(t_0)}{dt} \in R(t_0, x(t_0)).$$

Finally the following lemma due to Fillipov [7] which we state without proof guarantees the existence of an admissible control yielding the trajectory $x(t)$.///

1.10 Lemma: Let the vector function $f(x(t),u,t)=g(u,t)$ be jointly continuous in u and t, and suppose $\Omega = \{u/u(t) \in Q(t)\}$ is the set of admissible controls where $Q(t)$ is compact and u.s.c. with respect to inclusion (in t). Let the vector $f(x(t), u(t), t)$ describe a set $R(t)$ when $u(t)$ describes $Q(t)$. Then if $y(t)$ is a measurable vector function such that $y(t) \in R(t)$ there exists a measurable vector function $u(t) \in Q(t)$ such that $f(x(t), u(t), t) \equiv y(t)$ for almost all t.

We have now considered systems (0.1) for which $S_T(X^0)$ is closed and bounded. In general, however, $S_T(X^0)$ is not convex. We will now consider two ways of "making" it convex under additional assumptions. The following assumptions lead to the first and easiest approach.

A8. Suppose $f(x,u,t)$ is of the form

$$f_i(x,u,t) = \sum_{j=1}^{n} A_i^j(t)\, x + u_i \qquad i = 1,\ldots,n$$

where $A_i^j(t)$ is continuous for each i,j and suppose $f_0(x,u,t)$ is convex in (x,u) for all x, u, and t; i.e.,

$$f_0(\lambda x^1 + (1-\lambda)x^2, \lambda u^1 + (1-\lambda)u^2, t) \leq$$

$$\lambda f_0(x^1,u^1,t) + (1-\lambda)\, f_0(x^2,u^2,t) \text{ for } 0 \leq \lambda \leq 1,$$

where $u = (u_1,\ldots,u_n)$

A9. $\Omega = \{u(t) / u(t) \in Q(t)$, u measurable and bounded$\}$ where $Q(t)$ is closed, bounded and convex for all t and is upper semi-continuous as a function of t.

1.11 Theorem: Under assumptions A1, A2, A3, A4, A8, and A9, the set

$$S_T^+(X^0) = \{x / x_0 \geq x^t,\ x_1 = x_1^t, \ldots, x_n = x_n^t$$

$$\text{for some } x^t \in S_T(X^0)\}$$

is convex if X^0 is.

Remarks: Notice in A8 that none of the $f_i(x,u,t)$ $i = 1,\ldots,n$ depends on x_0. $S_T^+(X^0)$ is equivalent to $S_T(X^0)$ for purposes of minimization. In the proof below, T could be replaced by t^0 for any $0 \leq t^0 \leq T$ thereby establishing the convexity of $S_t^+(X^0)$.

Proof: (of theorem)

Let x^{1T} and $x^{2T} \epsilon S_T^+(X^0)$ then there exist by the definition of S_T^+ u^1, u^2, x^1, x^2 such that

$$\frac{dx^k}{dt} = f\ (x^k(t), u^k(t), t) \quad \text{a.e.} \quad 0 \leq t \leq T, \quad k = 1,2.$$

$$x^k(0) \epsilon X^0 \text{ and } x_0^k(T) \leq x_0^{kT}, \quad x_i^k(T) = x_i^{kT} \quad i = 1,\ldots,n, \quad k = 1,2.$$

By the convexity of X^0, $\lambda x^1(0) + (1-\lambda)\ x^2(0) \epsilon X^0$ and by the convexity of $Q(t)$ $u^\lambda(t) = \lambda u^1(t) + (1-\lambda)\ u^2(t)$ is an admissible control. Then we observe that

$$\frac{d}{dt}(\lambda x_i^1(t) + (1-\lambda)x_i^2(t)) = \lambda \frac{d}{dt} x_i^1(t) + (1-\lambda)\frac{dx_i^2}{dt}(t)$$

$$= \lambda\left[\sum A_i^j x_j^1(t) + u^1(t)\right] + (1-\lambda)\left[\sum A_i^j x_j(t) + u_i^2(t)\right]$$

$$= \sum A_i^j[\lambda x_j^1 + (1-\lambda)x_j^2] + \lambda u_i^1 + (1-\lambda)u_i^2 \quad i = 1,\ldots,n.$$

Also $\frac{dx_0}{dt} = f_0\ (x(t), u^\lambda(t), t)$ is equivalent to

$$x_0(t) = x_0(0) + \int_0^t f_0(x(t), u^\lambda(t), t)dt.$$

By the convexity of f_0 we have that

$$(1.12) \quad \int_0^T f_0(\lambda x^1(t) + (1-\lambda)x^2(t), u^\lambda(t))$$

$$\leq \int_0^T \lambda f_0(x^1(t), u^1(t), t)dt + \int_0^T (1-\lambda)f_0(x^2(t), u^2(t), t)dt$$

Then $x^\lambda(t)$ given by

$$x_i^\lambda(t) = \lambda x_i + (1-\lambda)x_i^2 \qquad i = 1,\ldots,m$$

$$x_0^\lambda(t) = \lambda x_0^1(0) + (1-\lambda)x_0^2(0) + \int_0^t f_0(x_1^\lambda,\ldots,x_n^\lambda,u^\lambda,s)ds$$

is a solution of the differential equation for the control

u^{λ} and initial condition $x(0) = \lambda x^1(0) + (1-\lambda)x^2(0)$. We also have by 1.12 that $x_0^{\lambda} \leq \lambda x_0^1(T) + (1-\lambda)x_0^2(T)$. Thus by construction $\lambda x^{1\,T} + (1-\lambda)x^{2\,T} \in S_T^+(X^0)$.///

The second approach to the convexity problem is due to Halkin [10]. We merely outline the development here, and for convenience we assume that $X^0 = \{x^0\}$. The appropriate assumptions to replace A8 and A9 are the following:

A 10. $\Omega = \{u / u(t) \in Q,\ 0 \leq t \leq T,$ u bounded and measurable$\}$ for some set $Q \subset E^r$.

A 11. For any trajectory x(t) satisfying (0.1) there exists $\epsilon > 0$ such that f and $\nabla_x f$ are defined, uniformly equicontinuous with respect to x, and uniformly bounded for all $x,u,t \in N(x,\xi) \times Q^*$ where

$$N(x,\epsilon) = \{(\bar{x},\bar{t}) \ / \ \| \bar{x} - x(t) \|^2 + | \bar{t} - t |^2 \leq \epsilon^2,$$
$$0 \leq t \leq T\}$$ where Q^* is any bounded subset of Q.

Let Y(t;u) be the matrix solution of the adjoint equation

$$\frac{dY}{dt} = -Y \ \nabla_x f \Big|_{\substack{x = x(t;u) \\ u = u(t)}} \quad ; Y(T) = I,$$

where I is the (n+1) x (n+1) identity matrix and x(t;u) denotes the solution of

$$\frac{dx}{dt} = f(x,u,t) \quad \text{a.e.} \quad 0 \leq t \leq T \quad x(0) = x^0.$$

Then we define y(t;u,v) by

(1.13) $y(t;u,v) = Y(t;v)(x(t;u) - x(t;v))$

for $0 \leq t \leq T$ and an approximation $y^+(t;u,v)$ to $y(t;u,v)$ by

(1.14) $y^+(t;u,v) = \int_0^t Y(\tau,v)(f(x(\tau;v),u(\tau),\tau) -$

$$f(x(\tau;v),v(\tau),\tau)d\tau$$

Roughly speaking, if at time t, one imposes the condition $x(t) = x(t;u)$ and then uses the control v for the remainder of the trajectory, i.e., to time T, one arrives at $x(T;v) + y(t;u,v)$ at least to a linear approximation. In even rougher terms, $y^+(t;u,v)$ is the best approximation to $y(t;u,v)$ that an observer traveling along $x(t;v)$ can make knowing only the relevant quantities evaluated for x on the trajectory $x(t;v)$.

Finally, for fixed control v we introduce the two sets

$H_t(v) = \{y(t;u,v) \ / \ u \in \Omega\}$

$H_t^+(v) = \{y^+(t;u,v) \ / \ u \in \Omega\}$.

Clearly,

$$H_T(v) = \{\alpha - x(T;v) \ / \ \alpha \in S_T(x^0)\}$$

and $H_t(v)$ is convex iff $S_t(x^0)$ is. In general, of course, neither are, but the principle results of [10] are:

1.15 <u>Theorem</u>: <u>Under assumption A2, A5, A 10 and A 11, $H_t^+(v)$ is convex for $0 \leq t \leq T$ for any admissible control.</u>

<u>Proof</u>: [10; p.94].

1.16 <u>Theorem</u>: <u>Under the above hypotheses, if $y = 0$ is a boundary point of $H_T(v)$, it is a boundary point of $H_T^+(v)$.</u>

<u>Proof</u>: [10; p.103].

In the next section we give a derivation of Pontryagins Maximum Principle [15] based on the system assumed in Theorem 1.11 since that will be sufficient for our purposes. The derivation based on A 10, A 11 and Theorems 1.15 and 1.16 is very similar.

2. <u>The Maximum Principle</u>: The problem we consider initially is the following:

$$
\text{(2.1)} \qquad
\begin{aligned}
&\text{Minimize} && x_0(T) \\
&\text{subject to} && x(0) = x^0 \\
& && x_i(T) = x_i{}^T \qquad i = 1,\ldots,n \\
& && \frac{dx}{dt} = f(x,u,t) \\
& && u \in \Omega
\end{aligned}
$$

under the assumptions A2, A4, A5, A7, A8, A9 of the previous section.

<u>**Remarks**</u>: $f(x,u,t)$ is, of course, of the very special form

$$f_i = \sum_{j=1}^{n} A_i^j(t)\, x_j + u_i \qquad i = 1,\ldots,m$$

f_0 convex in (x,u) but for ease of notation we will write it in the general form. Also in A7 we must assume the convexity of $R^+(x,t) = \{x/\, x_0 \geq f_0,\ x_1 = f_1,\ldots,x_n = f_n;\ f \in R(x,t)\}$ rather than $R(x,t)$ but this does not alter our problem.

Under these hypotheses, $S_T^+(x^0)$ is bounded, closed and convex. Thus (2.1) is reduced to a programming problem in fundamental form (II.1.1).

$$
\text{(2.2)} \qquad
\begin{aligned}
&\text{Max} && Z \\
&\text{s.t.} && U_0 Z + P = 0 \\
& && P \in \mathcal{X}
\end{aligned}
$$

Where $\chi = \{P \ / \ P = x - x^T,\ x \in S_T^+(x^0)\}$, $x^T = (0, x_1^T, \ldots, x_n^T)$.

The first two conditions of (2.1) can be generalized by allowing

$x(0) \in X^0$, $(x_1(T), \ldots, x_n(T)) \in X^T$ where X^0 and X^T are compact convex sets. Note $X^0 \subset E^{m+1}$ while $X^T \subset E^m$. In this case, (2.2) becomes

$$\text{(2.3)} \qquad \text{Max} \quad Z$$
$$\text{s.t.} \quad U_0 Z + P + Q = 0$$
$$P \in \chi', \quad Q = (0, x_1, \ldots, x_n) \text{ for some}$$

$(x_1, \ldots, x_n) \in X^T$, and $\chi' = \{P \ / \ P \in S_T(X^0)\}$. We will, however, confine our discussion to (2.1) and (2.2); the generalization to (2.3) is quite simple.

We assume that (2.2) is regular. Then in theory we could apply the generalized simplex method to obtain the optimal point P^* and the dual variables π, μ. This, however, does not give any information about optimal trajectories $x^*(t)$ or optimal controls $u^*(t)$. To help us do this, we consider the following simple lemma:

2.4 Lemma: If a solution $\check{x}(t)$, of $\frac{dx}{dt} = f(x,u,t)$, $u \in \Omega$, $x(0) \in X^0$ belongs to the boundary of $S_T(X^0)$, then $\check{x}(t)$ belongs to the boundary of $S_t(X^0)$ for $0 \leq t \leq T$.

Proof: Suppose $\check{x}(t_1)$ is an interior point of $S_{t_1}(X^0)$. Let $M_{t_1}: E^{n+1} \to E^{n+1}$, be the map which takes points $x = x(t_1)$ of solutions into the value of the solution at time T.

$x(T)$, with the control changed to the same control as for $\tilde{x}$. Let U be any neighborhood of $\tilde{x}(t_1)$ contained in $S_{t_1}(X^0)$ and consider $V = M_{t_1}(U)$. Clearly $V \subset S_T(X^0)$. By A.1.7 $M_{t_1}^{-1}$ is continuous, hence V is a neighborhood of $\tilde{x}(T)$ contained in $S_T(X^0)$ which is a contradiction since $\tilde{x}(T)$ is a boundary point of $S_T(X^0)$.

We wish now to find, for $0 \leq t \leq T$, a supporting hyperplane defined by $\Pi(t)$ to $S_t^+(X^0)$ containing $x^*(t)$ where x^* ix an optimal trajectory. Clearly we want to take $\Pi(T) = \pi$ where π is optimal for the dual of (2.2). We now seek to determine $\Pi(t)$ for other times t.

At time T we have $\Pi(T)[x^*(T) - x] \geq 0$ for all $x \in S_T(X^0)$. We now consider $x^*(t) - x(t;\ T,x,u^*)$ where $x(t;\ T,x,u^*)$ denotes the solution of (0.1) satisfying $x(T;\ T,x,u^*) = x$ where the control is $u = u^*$. Making use of (B.3.1) we can represent this difference by means of solutions to the variational equation, i.e.,

$$x^*(t) - x(t;\ T,x,u^*) = ||x^*(T) - x||\ h(t) + o(||x^*(T) - x||)$$

where $h(t)$ satisfies

$$\text{(2.5)} \qquad \frac{dh(t)}{dt} = \nabla f \Big|_{\substack{x = x^*(t) \\ u = u^*}} \cdot h(t)$$

where ∇f is to be evaluated along $x^*(t), u^*(t)$,

and (2.6) $h(T) = \frac{x^*(T) - x}{||x^*(T) - x||}$. In order for $x^*(t)$ to be optimal there must be no vector x^1 and time t_1 such that

$$x^1 = x^*(t_1) + h(t_1)||x^1 - x^*(t_1)|| + o||x^1 - x^*(t_1)|| \epsilon S^+_{t_1}(x^0)$$

and $\Pi(T)\, h(T) < 0$ for some solution of (2.5). For if this were the case by taking x^1 close enough to $x^*(t_1)$ on the line joining them and by changing the control which achieves x^1 to u^* on $t_1 \leq t \leq T$ we would obtain a point, x^T, in $S_T(x^0)$ such that $\Pi(x^T - x^*(T)) = \Pi(T)\ (x^T - x^*(T)) < 0$ which would be a contradiction.

To help us find points, x^1, in $S_t(x^0)$ we consider the adjoint equation to (2.5)

$$(2.7) \quad \frac{d\Pi(t)}{dt} = -\Pi(t)\ \nabla f\Big|_{\substack{x = x^*(t) \\ u = u^*}}$$

Solutions of (2.7) have the property that for any solution $h(t)$ of (2.5) we have

$$\begin{aligned} \frac{d}{dt}\left[\Pi(t)\ h(t)\right] &= \left[\frac{d}{dt}\ \Pi(t)\right] h(t) + \Pi(t)\ \frac{d}{dt}\ h(t) \\ &= -\Pi(t)\ \nabla f h(t) + \Pi(t)\ \nabla f\, h(t) \\ &= 0. \end{aligned}$$

As intimated by the notation, the solution, $\Pi(t)$, of (2.7) with the terminal condition $\Pi(T) = \pi$ is the supporting hyperplane ··ant. We formalize this as the following theorem:

2.8 Theorem: In order for $x^*(t) = x(t;0,x^0,u^*)$ to be an optimal solution of (2.1), there must exist a solution, $\Pi^*(t)$, of 2.7 such that

$$\min_{x \in S_t(x^0)} \Pi^*(t)x = \Pi^*(t)\, x^*(t) \text{ for } 0 \leq t \leq T.$$

Proof: Suppose $\Pi(t_1)\ [x^1 - x^*(t_1)] < 0$ for $x^1 \in S_{t_1}(x^0)$, then $\lambda x^1 + (1-\lambda)\, x^*(t_1) \in S^+_{t_1}(x^0)$ for $0 \leq \lambda \leq 1$. Let $h(t)$ be the solution of (2.7) satisfying $h(t_1) = \dfrac{x^1 - x^*(t_1)}{||\ x^1 - x^*(t_1)\ ||}$.

Then $\Pi(t_1)\ h(t_1) = \Pi(T)\ h(T) = \pi\ h(T) \quad 0$. Furthermore, $\pi[x(T;\ t^1,\ x^1,\ u^*) - x(T;\ t^1,\ x^*(t_1),\ u^*)] =$
$\pi\{h(T)||x^1 - x^*(t_1)|| + o(||x^1 - x^*(t_1)||)\} < 0$
for x^1 sufficiently close to $x^*(t_1)$ along the line joining them. This is a contradiction hence $\Pi(t)\ [x - x^*(t)] \geq 0$ for all t and $x \in S^+_t(x^0)$ and hence for $x\ S_t(x^0)$.///

Remark: Theorem 2.8 holds whether (2.2) is regular or not.

2.9 Corollary: The affine dimension of $S_t(x^0)$ is non-decreasing in t.

Proof: Suppose $\Pi^j(T)(x - x^*(T)) = 0$ for all $x \in S^+_T(x^0)$ $j = 1,\ldots,r$. Then $\Pi^j(t)(x - x^*(t)) = 0$ for all $x \in S^+_t(x^0)$ $j = 1,\ldots,r$ for $t \leq T$. Where $S^+_t(x^0)$ denotes the convex hull of $S_t(x^0)$. Finally, if $\Pi^j(T)(x - x^*(T)) = 0$ for all $x \in S_T(x^0)$ but there exists $x^1 \in S^+_T(x^0)$ such that $\Pi^j(T)(x^1 - x^*(T) \neq 0$ we easily see that $S_T(x^0)$ must be convex. Then we apply the first argument to S_T, S_t rather than to S^+_T, S_t.///

Finally we come to the celebrated maximum principle of Pontryagin (here because of a choice of sign it become a "minimum" principle). First we must impose a further condition on the class of admissible controls.

<u>Definition</u>: $Q(t)$ is said to admit <u>constant</u> <u>variations</u> at t if for every $u \in Q(t)$ and $\epsilon > 0$, there exists $u^1, \delta > 0$, such that $\| u^1 - u \| < \epsilon$ and $u^1 \in Q(t^1)$ for all t^1 such that $t^1 \in I$ where I is either the interval $[t, t+\delta]$ or $[t-\delta, t]$.

<u>Examples</u>: $Q(t) = U$ a constant set, or $Q(t) = U(t)$, a set function monotonic in t are examples of sets admitting constant variations.

2.10 <u>Theorem</u>: <u>Assume that $Q(t)$ admits constant variations for all t; and that $u^*(t)$, $0 \leq t \leq T$ is an admissible control such that the corresponding trajectory $x^*(t)$ satisfies</u>

$$x^*(0) = x^0$$

$$x_i^*(T) = x_i^T \qquad i = 1,\ldots,n$$

$$\frac{dx^*}{dt} = f(x^*, u^*, t) \qquad \text{in } t.$$

<u>Then in order for $u^*(t)$ and $x^*(t)$ to be optimal, it is necessary that there exist a solution, $\Pi(t)$, of (2.7) such that</u>:

$$\text{(2.11)} \quad H(\Pi(t), x^*(t), u^*(t)) = \min_{u \in Q(t)} H(\Pi(t), x^*(t), u)$$

<u>almost everywhere in t for $0 \leq t \leq T$, where $H(\Pi(t), x(t), u(t)) = (\Pi(t) \;\; f(x(t), u(t), t))$</u>.

<u>Furthermore, $\Pi_0(t) \geq 0$ is constant.</u>

<u>Proof</u>: We use the same $\Pi(t)$ as in theorem 2.8.

Applying Theorem 2.8 we have:

$$\Pi(t_1+\Delta t)\left\{\int_{t_1}^{t_1+\Delta t}[f(x(\tau),u,\tau)-f(x^*(\tau),u^*(\tau),\tau)]d\tau\right\}\geq 0$$

for any control u.

For almost all t_1

$$(2.12)\qquad \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{t_1}^{t_1+\Delta t} f(x^*(\tau),\ u^*(\tau),\ \tau)d\tau$$

$$= f(x^*(t_1),u^*(t_1),\ t_1)\quad \text{see [13, p.255].}$$

Suppose (2.11) does not hold almost everywhere. Then there is a point t_1 such that (2.11) does not hold, but (2.12) does. Let u^1 be such that

$$\Pi(t_1)[f(x^*(t_1),\ u^1,\ t_1) - f(x^*(t_1),\ u^*(t_1),\ t_1)] < 0.$$

Since $Q(t)$ admits constant variations, and by the continuity of $f(x,u,t)$ in its arguments there exists $\delta,\delta_1 > 0$, and u such that

$$\Pi(t)[f(x(t),u,t) - f(x^*(t_1),u^*(t),\ t_1)] < -\delta/2$$

for $t_1 \leq t \leq t_1 + \delta_1$

(or $t_1 + \delta_1 \leq t \leq t_1$ in which case the modifications will be left to the reader).

Then using (2.12) we have

$$\int_{t_1}^{t_1+\Delta t}[f(x(t),u,t) - f(x(t),u^*(t),t)]dt$$

$$= [f(x^*(t_1),u,t_1) - f(x^*(t_1),u^*(t_1),t_1)]\Delta t + o(\Delta t).$$

Multiplying both sides by $\frac{\Pi(t_1)}{\Delta t}$ and letting $\Delta t \to 0$ yields

$$\lim_{\Delta t \to 0} \frac{\Pi(t_1+\Delta t)}{\Delta t} \left[\int_{t_1}^{t_1+\Delta t} [f(x(t),u,\tau) - f(x^*(\tau),u^*(\tau),\tau)]dt\right] < 0$$

which is a contradiction.

That $\Pi_0(t)$ is constant follows from the fact that

$$\frac{\partial f_i}{\partial x_0} = 0 \qquad i = 0,\ldots,n.///$$

We close this section by indicating how Halkin proves the maximum principle in [10]. Clearly for an optimal trajectory $x^*(t)$ with corresponding control $u^*(t)$ we have that $x^*(T)$ is a boundary point of $S_T(x^0)$. This is equivalent to $y = 0$ being a boundary point to $H_T(u^*)$. By Theorem 1.16, $y = 0$ is a boundary point to $H_T^+(u^*)$ which by Theorem 1.15 is convex. Let π denote the inward pointing normal to a supporting hyperplane of $H_T^+(u^*)$ at $y = 0$. By the definition of $H_T^+(u^*)$ we obtain

$$(2.13) \quad 0 \leq \pi \int_0^T Y(\tau,u^*)\ (f(x^*(\tau),\ u(\tau),\tau) - f(x^*(\tau),\ u^*(\tau),\tau)dt$$

for all admissible control functions u. Let $\Pi(t) = \pi Y(t,u^*)$ which satisfies (2.7). Pointwise the integrand of (2.13) mu ^ be non-negative almost everywhere since Ω by A 10 obviously admits constant variations, and the previous argument goes through. Of course in this case $\Pi_0(t)$ need not be a constant.

3. Transversality Conditions: We now return to the problem (2.3) and show how in a very easy way we can obtain transversality conditions for the initial and terminal points.

3.1 Theorem: For (2.3), the problem with variable endpoints, theorem 2.10 is valid and moreover we can choose $\Pi(t)$ such that $\Pi(T)\,[x - x^*(T)] \leq 0$, for all x such that $x = (x_0^*(T), x_1, \ldots, x_n)$ where $(x_1, \ldots, x_n) \in X^T$. $\Pi(0)[x - x^*(0)] \geq 0$, for all $x \in X^0$, whenever both X^0, and X^T are compact and convex.

Proof: Given an optimal trajectory it must solve the fixed endpoint problem for $x^0 = x^*(0)$, $x^T = x^*(T)$ hence 2.10 is valid. $S_0(X^0)$ is X^0 hence $\Pi(0)[x - x^*(0)] \geq 0$, using lemma 2.4. $\Pi(T)[x - x^*(T)] = \pi[x - x^*(T)]$ follows from II.5.5.///

In order to be more specific about how the simplex method can be applied to optimal theory, we treat, in the next section, the special case of linear control problems.

4. Linear Optimal Control: We consider here the following problem:

$$(4.1) \qquad \text{Min} \quad x_0(T)$$

$$\text{s.t.} \quad \frac{dx(t)}{dt} = A(t)\, x(t) + u(t)$$

$$x(0) \in X^0$$

$$(x_1(T), \ldots, x_n(T) \in X^T$$

$$u \in \Omega = \{u : u \text{ is measurable, and } u(t) \in Q\ (t)\}$$

Where $A(t)$ is continuous on $[0,T]$, $Q(t)$ is convex, compact for all t, $0 \leq t \leq T$, and as a set function is upper semi-continuous in t. Further, we assume that $Q(t)$ admits constant variations. X^0, and X^T are assumed to be convex and compact.

Remarks: $f(x,u,t) = A(t)x + u$ is continuously differentiable with respect to x and is continuous with respect to t and u. By the results of the previous section, $S_t(x^0)$ is closed, bounded and convex for each t, $0 \leq t \leq T$. The conditions for Fillippov's Lemma 1.10 are also satisfied. Moreover, we are aided because we have an explicit representation for the solutions of 4.1 for various $u \in \Omega$ (B.2.5).

$$(4.2) \quad x(t) = Y(t)\, x(0) + \int_0^t Y(t)\, Y^{-1}(s)\, u(s)\, ds, \text{ where}$$

Y satisfies the matrix equation

$$(4.3) \quad \frac{dY(t)}{dt} = A(t)\, Y(t) \text{ with initial condition}$$

$$Y(0) = I, \text{ the identity matrix.}$$

If $x^*(t), u^*(t)$ are optimal for (4.1) then there exists a π such that

$\pi(x(T) - x^*(T)) \geq 0$ for all $x(T) \in S_T(x^0)$ Using (4.2) we obtain

$$\pi\left[\int_0^T Y(T)\, Y^{-1}(s)\, (u(s) - u^*(s))ds\right] \geq 0 .$$

Letting $\Pi(s) = \pi Y(T)\, Y^{-1}(s)$ we observe (B.3.6) that $\Pi(s)$ is a solution of the adjoint equation for (4.1). This, of course, is the same solution as obtained in Theorem 2.8. For (4.1) the Hamiltonian $H = \Pi(t)\, f\,(t) = \Pi(t)[A(t)\, x(t) + u(t)]$ and $H(x,\Pi,u^*) - H(x,\Pi,u) = \Pi(t)[u(t) - u^*(t)]$. Hence the maximum principle 2.10 specialized to the linear case becomes:

4.4 Theorem: Suppose that $u^*(t)$, $0 \leq t \leq T$ is an admissible control such that the corresponding trajectory $x^*(t)$ satisfies

$$x^*(0) = x^0$$

$$x^*_i(T) = x_i^T \qquad i = 1,\ldots,n$$

$$\frac{dx^*(t)}{dt} = A(t)\, x^*(t) + u^*(t) \quad \text{a.e. in } t.$$

Then in order for u^* and x^* to be optimal it is necessary that there exist a solution $\Pi(t)$ of $\frac{d\Pi(t)}{dt} = -\Pi(t)\, A(t)$ such that $0 \leq H(\Pi(t),x^*(t),u) - H(\Pi(t),x^*(t),u^*(t))$.

$$= \Pi(t)[u - u^*(t)] \text{ for all } u \in Q(t)$$

for almost all t. Furthermore $\Pi_0(T) \geq 0$.

If we knew $\Pi(T) = \pi$, we could determine $\Pi(t) = \pi Y(T)\, Y^{-1}(t)$ and then try to determine u^* by means of

$$\Pi(t)\, u^*(t) = \min_{u \in Q(t)} \Pi(t)\, u. \tag{4.5}$$

If (4.5) determines u^* uniquely almost everywhere we could then solve for $u^*(t)$. Conditions which guarantee this, as well as stronger results for special cases of (4.1) can be found in [11], [15; Ch.3]. In the next section we investigate in some detail the application of the generalized simplex method to a rather general linear optimal control problem.

5. The equivalent generalized program: We formulate (4.1) with varying endpoints as a generalized program.

$$
\begin{array}{lllr}
 & \text{Max} & Z & \\
(5.1) & \text{s.t.} & U_0 Z + \lambda_1 P^{\Omega} + \mu P^{0} + \nu P^{T} & = 0 \\
 & & \lambda_1 & = 1 \\
 & & \lambda_2 & = 1 \\
 & & \lambda_3 & = 1 \\
 & & (\lambda_i \geq 0) & i = 1,2,3
\end{array}
$$

$$P^{\Omega} \epsilon \mathcal{K}^{\Omega}, \quad P^{0} \epsilon \mathcal{K}^{0}, \quad P^{T} \epsilon \mathcal{K}^{T}$$

where

$$\mathcal{K}^{\Omega} = \{ \int_0^T Y(T)\, Y^{-1}(s)\, u(s)\, ds \;/\; u \epsilon \Omega \}$$

$$\mathcal{K}^{0} = \{ Y(T)\, x^{0} \;/\; x^{0} \epsilon X^{0} \}$$

$\mathcal{K}^{T} = \{ (0, -x^{T}) / x^{T} \epsilon X^{T} \}$. That (5.1) and (4.1) are equivalent follows directly from (4.2). $\mathcal{K} = \mathcal{K}^{\Omega} \oplus \mathcal{K}^{0} \oplus \mathcal{K}^{T}$ is closed, bounded and convex. If $\mathcal{K}$ is regular the generalized simplex method converges to the optimal solution. Suppose at step k we have prices $\pi^{k}, \mu_{\Omega}{}^{k}, \mu_{0}{}^{k}, \mu_{T}{}^{k}$ then the subproblem is

$$\Delta^{0} = \text{Min}\, \{ \pi^{k} P^{0} \;/\; P^{0} \epsilon \mathcal{K}^{0} \} + \mu_{0}{}^{k}$$

$$\Delta^{T} = \text{Min}\, \{ \pi^{k} P^{T} \;/\; P^{T} \epsilon \mathcal{K}^{T} \} + \mu_{T}{}^{k}$$

$$\Delta^{\Omega} = \text{Min}\, \{ \pi^{k} P^{\Omega} \;/\; P^{\Omega} \epsilon \mathcal{K}^{\Omega} \} + \mu_{\Omega}{}^{k}$$

$$\Delta = \min\, (\Delta^{0}, \Delta^{T}, \Delta^{\Omega}).$$

The determination of Δ^{0} and Δ^{T} involves solving two convex programs and, for example, if X^{0}, and X^{T} are polyhedral, then the

determination is simply a linear program. Now we investigate in more detail the determination of Δ^{Ω}:

$$(5.2)\quad \Delta^{\Omega}-\mu_{\Omega}{}^{k} = \mathrm{Min}\ \{\pi^{k}P^{\Omega} \,/\, P^{\Omega}\epsilon\mathcal{K}^{\Omega}\}$$

$$= \underset{u\epsilon\Omega}{\mathrm{Min}}\ \{\pi^{k} \int_{0}^{T} Y(T)\ Y^{-1}(s)\ u(s)\ ds\}$$

$$= \underset{u\epsilon\Omega}{\mathrm{Min}}\ \{ \int_{0}^{T} \Pi^{k}_{(s)}\ u(s)\ ds\} \text{ where } \Pi^{k}(s) = \pi^{k}Y(T)\,Y^{-1}(s)$$

One solves for the control in the above, and then introduces the corresponding value of the integral $\int_{0}^{T} Y(T)\ Y^{-1}(s)\,u(s)\,ds$ and into the master problem if $\Delta = \Delta^{\Omega}$. In one particular case, the calculations involved are relatively easily carried out. That is for the so called "bang-bang" problem. Here

$\Omega = \{u : u(t) = B(t)\ v(t),\ |v_{i}(t)| \leq 1 \quad u \text{ measurable}\}$

where $B(t)$ is a continuous $n \times r$ matrix, and $v(t)$ an r vector. Since the optimal control can always be taken on the boundary of $Q(t)$ $|v_{i}(t)| = 1$ can be assumed without loss of generality. Hence the name "bang-bang". In this case, the optimizing u is given by $u_{i}(t) = \mathrm{sgn}[\Pi(s)\ B(s)]_{i}$ where "sgn" is $+1$ or -1 depending o. whether its argument is positive or non-positive. In the even more particular case where $A(t) = A$, a constant matrix, then

$$\Pi(s) = \pi^{k}e^{A(T-s)}$$

$$= \pi^{k} \sum_{j=0}^{\infty} \frac{A^{j}(T-s)^{j}}{j}$$

$$= \pi^k \sum_{j=0}^{m} F^j(T-s)\, e^{\lambda j(T-s)}$$

where λ_j, $j = 1,\ldots,m$ are the distinct eigenvalues of $-A^{Tr}$ and $F_i^{\,j}$ is a polynomial in (T-s) of order less than the multiplicity of λ_j [21; App. D]. If $-A^{Tr}$ has distinct eigenvalues F is a constant matrix. So in this special case, the "pricing out" for Δ^{Ω} can be accomplished more or less analytically. In general, however, the adjoint matrix equation must be solved numerically, say by discretization, before the algorithm starts. In the next section we show how the discretized problem can be solved efficiently using the variant of the simplex method outlined in section I.3.

6. <u>A computational method</u>: We first give the formal details of the computational method suggested by G. B. Dantzig. We then indicate how the method can be made more efficient by use of techniques in numerical analysis.

The problem we consider is the following:

$$(6.1)\qquad \begin{aligned} &\text{Min} && x_0(T) \\ &\text{s.t.} && \frac{dx(t)}{dt} = A(t)\, x(t) + u(t) \\ &&& x(0) \in X^0 \\ &&& (x_1(T), \ldots, x_n(T)) \in X^T \\ &&& u \in \Omega = \{u(t) : u(t) \in U(t),\ U(t) \text{ a closed bounded convex set, } u \text{ measurable and bounded}\} \\ &&& X^0,\ X^T \text{ compact and convex.} \end{aligned}$$

The first step is to replace (6.1) by a discrete analog. For convenience, we will divide the time interval $[0,T]$ into K equal parts, i.e., $t_0 = 0$, $t_1 = \Delta t$, $t_2 = 2\Delta t, \ldots, t_K = K\Delta t = T$. For large enough K we can replace $\frac{dx}{dt}(t_j)$ by $\frac{\Delta x^j}{\Delta t}$, where $\Delta x^j = x(t_{j+1}) - x(t_j)$. If we denote $x(t_j)$ by x^j, $u(t_j)$ by u^j, $A(t_j)$ by A^j and $U(t_j)$ by U^j for $j = 0, ., \ldots, K$, we can replace (6.1) by

$$(6.2)\qquad \begin{aligned} &\text{Min} && x_0^K \\ &\text{s.t.} && x^{j-1} = \bar{A}^j x^j + u^j \Delta t \qquad j = 0, \ldots, K-1 \\ &&& x^0 \in X^0 \\ &&& (x_1^K, \ldots, x_n^K) \in X^T \\ &&& u^j \in U(t_j) \qquad j = 0, \ldots, K, \end{aligned}$$

where $\bar{A}^j = [I + A^j \Delta t]$. We can solve (6.2) in terms of u^j by successive substitutions:

$$x^K = [\bar{A}^{K-1}\bar{A}^{K-2}\ldots\bar{A}^0]x^0 + \sum_{i=0}^{K-1} \Delta t[\bar{A}^{K-1}\bar{A}^{K-2}\ldots\bar{A}^{i+1}]u^i$$

where we make the convention that for $i = K-1$ $[A^{K-1}A^{K-2}\ldots A^{i+1}] = I$, the identity matrix. Now letting $w^i = (A^{K-2}\ldots A^{i+1})u^i$ and $A^* = [\bar{A}^{K-1}\bar{A}^{K-2}\ldots\bar{A}^0]$ we obtain the generalized program

(6.3) Max Z

$$\begin{aligned} \text{s.t.} \quad U_0 Z + \nu_T P^T + \nu_0 P^0 + \sum_{i=0}^{K-1} \lambda_i w^i &= 0 \\ \nu_T &= 1 \\ \nu_0 &= 1 \\ \lambda_i &= 1 \\ (\nu_T, \nu_0, \lambda_i) &\geq 0 \end{aligned}$$

$$P^T \in \{P / P_0 = 0, (P_1, \ldots, P_n) \in X^T\},$$

$$P^0 \in \{P : P = A^* x, x \in X^0\}$$

$$w^i = \{w^i : w^i = [A^{K-1}A^{K-2}\ldots A^{i+1}]u, u \in U^i\}$$

We can then use the generalized programming algorithm to solve this. Moreover, (6.3) has the structure described in I.3, hence the generalized upper bounding techniques can be applied. This has two ramifications. One is that the computation involved in this approach is relatively insensitive to increases in K which determines the accuracy

of the discrete approximation. The other point of interest is that Theorem I.3.2 guarantees that at most $n+1$ values of w^1 (and hence u^1) cannot be taken as extreme points of their respective domains (see Theorem 4.4).

Remarks: The method described in this section is, practically speaking, no different from the approach described in the previous section. A^* is simply $Y(T)$ calculated by the method of finite differences, similarly $[\bar{A}^{K-1}\bar{A}^{K-2}\ldots\bar{A}^{i+1}]$ is the finite difference approximation to $Y(T)\ Y^{-1}(t_i)$, so the format given in this section is simply the implementation of the previous algorithm in a form usable on a digital computer. Existing linear programming codes can easily be modified to solve problems of the form (6.3) rapidly. Of course, few numerical analysts would go about solving (4.3) by utilizing the first order difference approximation for $\frac{dx}{dt}$. For fixed u, this approach leads to errors of the order of Δt [17; Sect. 2.24]. Other finite difference methods such as the Runge-Kutta method can yield accuracies on the order of $(\Delta t)^4$. It is an easy exercise to see that most of these numerical techniques lead to mathematical programs of the same form as (6.3), in particular the Runge-Kutta method yields a problem of that form which contains K extra variables.

ACKNOWLEDGMENTS

The frequent appearance of the symbol [L.P.E.] referring to Linear Programming and Extensions by George B. Dantzig [3] is only one indication of the extent of the influence of Professor Dantzig on this report. The origins of this study can all be found in his work. There are essentially two ways of interpreting mathematical programming problems geometrically. One is by considering the problem in the space of the independent variables (decision variables, activity levels, controls), the other is to consider the problem in the space of the dependent variables (state variables, activities, outputs). The second approach is the one followed here and was actually used by Dantzig when he first proposed the simplex method [L.P.E., Section 7-3]. In this geometry, the simplex method (at least formally) can easily be modified to solve convex programming problems [L.P.E., Ch. 24]. Dantzig, in 1960, gave a convergence proof for this algorithm. What I have termed the generalized simplex method is essentially that algorithm in a slightly more general setting. Dantzig arrived at his convex programming algorithm as a special case of the generalized linear program [L.P.E., Ch. 22]. In this report we reverse the process and show that generalized programs, as well as linear programming, the decomposition algorithm, and many problems in control theory are special cases of the convex programming problem.

A well known variant of the simplex method, also due to Dantzig [L.P.E., Ch. 18], is the upper bounding technique for solving linear programs with bounded independent variables. This was generalized in [4,5] to solve certain industrial scheduling problems. In April of 1964, Dantzig combined this generalized upper bounding technique with the ideas of a generalized program to propose an efficient method of solving linear control problems. Chapter III of this report is essentially a development of that suggestion. A similar approach can be found in [18, 19, 20].

My contribution, other than whatever errors there might be, is, I hope, twofold. One is to give proofs justifying certain formal procedures, such as taking duals, and applying the simplex method to the generalized program and to control problems. The second aim was to put all the special methods and problems of mathematical programming mentioned here into a general framework.

I would like to acknowledge the help of the faculty and fellow students who have sacrificed their time to give comments or simply listen to me expound on this report. I am also grateful to the Operations Research Center at the University of California and the RAND Corporation for their assistance and for providing an atmosphere of intellectual stimulation these past few years. Finally I would like to compliment Miss Susan Arbuckle for her courage and skill in the typing of this report in face of some very formidable notation.

Appendix A: Affine Geometry and the Theory of Convex Sets

1. Affine Geometry

We consider the group of affine transformations on the vector space E^m.

1.1 Definitions: $y = A(x) = Tx + t$ is an affine transformation from E^m into itself when T is a $m \times m$ non-singular matrix and t any m-vector. If $V \subset E^m$ is given by $V = \{y / y = A(x),\ x \in S\}$ where S is an affine transformation and S is a linear subspace of E^m of dimension k then V is said to be a linear variety of dimension k.

1.2 Remark: Clearly affine transformations carry linear varieties into linear varieties of the same dimension. For if V is a linear variety and $V = \{y = Tx + t / x \in S\}$ and W is given by $W = \{z = Ry + r / y \in V\}$ then $W = \{z = RTx + (Rt + r) / x \in S\}$ which is a linear variety since the product of two non-singular matrices is non-singular.

1.3 Definition: The p + 1 points $x^0, \ldots, x^p$ are affinely independent if $x^1 - x^0, \ldots, x^p - x^0_0$ are linearly independent as vectors. We note that affine independence does not depend on the choice of x^0.

Corresponding to the situation for linear spaces we wish to consider linear combinations of affinely independent points. But we wish to preserve the property that

(1.4) $A(\lambda_0 x^0 + \ldots + \lambda_p x^p) = \lambda_0 A(x^0) + \ldots + \lambda_p A(x^p)$

for all affine transformations A. This is not the case

for arbitrary affine transformations and linear combinations. To see this, let $y = \lambda_0 x^0 + \ldots + \lambda_p x^p$ and $A(x) = Tx + t$

then $A(y) = T(\lambda_0 x^0 + \ldots + \lambda_p x^p) + t$

$$
\begin{aligned}
&= \lambda_0 Tx^0 + \ldots + \lambda_p Tx^p + t \\
&= \sum \lambda_j Tx^j + \sum \lambda_j t - \sum \lambda_j t + t \\
&= \sum \lambda_j (Tx^j + t) + t(1 - \sum \lambda_j) \\
&= \sum \lambda_j A(x^j) + t(1 - \sum \lambda_j).
\end{aligned}
$$

Hence the property (1.4) holds only if $t = 0$ or $\sum \lambda_j = 1$. We impose the latter condition.

1.5 Definition: $y = \sum \lambda_j x^j$ is called a centroid of $x^1, \ldots, x^p$ if $\sum \lambda_j = 1$.

1.6 Theorem: The set V of all centroids of points $\{x^\alpha\}$, $\alpha \epsilon \mathcal{A}$ is a linear variety containing $\{x^\alpha\}$ and every linear variety containing $\{x^\alpha\}$ contains V. The linear variety V is said to be the linear variety "spanned" by $\{x^\alpha\}$.

Proof: Let S be the vector space spanned by $\{x^\alpha - x^{\alpha_0}\}$ for $\alpha \epsilon \mathcal{A}$, and fixed $\alpha_0 \epsilon \mathcal{A}$, and let $\overline{V}$ be the linear variety defined by $\overline{V} = \{z / z = I_m(z - x^{\alpha_0}) + x^{\alpha_0}\ (z - x^{\alpha_0}) \epsilon S\}$ where I_m is the mxm identity matrix. Assume now that

$Z = \sum \lambda_j x^j, \ \sum \lambda_j = 1.$ Then

$Z - x^{\alpha_0} = \sum \lambda_j x^j - \sum \lambda_j x^{\alpha_0} = \sum \lambda_j (x^j - x^{\alpha_0})$, and thus

$Z - x^{\alpha_0} \epsilon S$ for all $Z \epsilon V$, i.e., $V \ \overline{V}$. Conversely, let

$(z - x^{\alpha_0}) = \sum \lambda_j (x^j - x^0)$, then

$z = [1 - \sum \lambda_j] x^{\alpha_0} + \sum \lambda_j x^j$. Letting $\tilde{\lambda}_0 = 1 - \sum \lambda_j$ and

$\tilde{\lambda}_j = \lambda_j$, $j \neq 0$, we have $\sum \tilde{\lambda}_j = 1$. Hence $\overline{V} \subset V$ and therefore $V = \overline{V}$ a linear variety. If W is a linear variety containing $\{x^\alpha\}$ then $W = \{y / y = A(x) = Tx + t, x \in S'\}$ for some linear subspace S', and affine transformation A. But the transformation $A^{-1}_{(x)} = T^{-1}x - t$ is affine and maps W into S'.

Suppose

$y = \sum \lambda_j x^j$, $\sum \lambda_j = 1$ is a centroid of points in $\{x^\alpha\}$

Then $A^{-1}y = \sum \lambda_j A^{-1} x^j = \sum \lambda_j y^j$. But $A^{-1}x^j \in S$ hence

$A^{-1}y \in S'$. But $A(A^{-1}y) = y \in W$ finishing the proof.///

1.7 <u>Theorem:</u> <u>$x^0, \ldots, x^p$ are affinely independent iff every point y in the linear variety spanned by $x^0, \ldots, x^p$ has a unique representation as a centroid of $x^0, \ldots, x^p$.</u>

<u>Proof</u>: Suppose $x^0, \ldots, x^p$ are affinely independent and

$y = \sum \alpha_j x^j = \sum \beta_j x^j$, $\sum \alpha_j = \sum \beta_j = 1$. Then

$\sum (\alpha_j - \beta_j) x^j = 0$ and $\sum (\alpha_j - \beta_j) = 0$ and hence

$\sum_{j \neq 0} (\alpha_j - \beta_j)(x^j - x^0) = 0$ follows. Uniqueness means $\alpha_j = \beta_j$ must hold for $j = 0, 1, \ldots, p$. Suppose on the contrary, for some j_0 that $\alpha_{j_0} \neq \beta_{j_0}$. Without loss of generality, suppose $j_0 \neq 0$, then

$\sum(\alpha_j - \beta_j)(x^j - x^0)$ implies that

$(x^j - x^0)$ are linearly dependent contradicting the assumption that $x^0,\ldots,x^p$ are affinely independent.

Conversely, if $\sum \alpha_j(x^j - x^0) = 0$ and not all $\alpha_j = 0$,

then $(\sum \alpha_j)x^0 = \sum_{j=1}^{p} \alpha_j x^j$ which is not unique since

$x^0 = (1 - \sum \alpha_j)x^0 + \sum_{j=1}^{p} \alpha_j x^j$. On the other hand

$x^0 = \sum_{j=1}^{p} \frac{\alpha_j}{\sum \alpha_k} x^j$ if $\sum \alpha_j \neq 0$. If $\sum \alpha_j = 0$ then

$\sum \alpha_j x^j = \sum \alpha_j x^j - Z\alpha_j x^0 = 0.$

Pick any non-zero α_j, say α_k, then

$\alpha_k x_k = -\sum_{j\neq k} \alpha_j x^j$ where $\alpha_k = -\sum_{j\neq k} \alpha_j$.

We then obtain the same sort of contradiction as before.///

1.8 <u>Theorem</u>: <u>$x^0,\ldots,x^p$ are affinely independent iff the matrix</u> $X = \begin{bmatrix} x_1^0 & \cdots & x_1^p \\ x_2^0 & \cdots & x_2^p \\ \vdots & & \vdots \\ x_m^0 & \cdots & x_m^p \\ 1 & \cdots & 1 \end{bmatrix}$ <u>has rank $p+1$.</u>

<u>Proof</u>: Subtract the first column from the other columns.

This does not change the rank of X.

$$X' = \begin{bmatrix} x_1^0 & x_1^1 - x_1^0 & \cdots & x_1^p - x_1^0 \\ \vdots & \vdots & & \vdots \\ x_m^0 & x_m^1 - x_1^0 & \cdots & x_m^p - x_m^0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

$x^0, \ldots, x^p$ are affinely independent iff the last p columns of X' are linearly independent and clearly the first column is independent of the rest.///

For example, two points are affinely dependent iff they are the same point, three points are affinely dependent iff they are colinear, and four points are affinely dependent iff they are coplanar.

1.9 <u>Definition</u>: Let A be any subset of E^m. Its <u>linear dimension</u> ℓ.d.(A) is the dimension of the smallest subspace L_A containing A and its <u>affine dimension</u>, a.d.(A), is the dimension of the smallest linear variety, V_A, containing A.

1.10 <u>Lemma</u>: <u>a.d.(A) is equal to the maximum number of affinely independent points of A less one</u>.

<u>Proof</u>: V_A is simply the linear variety spanned by A (A.6). Suppose the maximum number of affinely independent points is p + 1 and let $x^0, \ldots, x^p$ be affinely independent. Consider the matrix defined in 1.8.

$$\begin{bmatrix} x_1^0 & x_1^1 - x_1^0 & \cdots & x_1^p - x_1^0 \\ \vdots & \vdots & & \vdots \\ x_m^0 & x_m^1 - x_m^0 & \cdots & x_m^p - x_m^0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

The last p columns are linearly independent and hence $y^j = x^j - x^0$ for $j = 1,\ldots,p$ span a p-dimensional vector space, and the affine transformation $y = x + x^0$ maps the linear subspace onto V_A. So V_A has affine dimension at least equal to p. Suppose there exists $z \in V_A$ such that $z - x^0$ is independent of the $\{y^j\}_1^p$ then

$$\begin{bmatrix} x_1^0 & \cdots & x_1^p & z_1 \\ \vdots & & \vdots & \vdots \\ x_m^0 & \cdots & x_m^p & z_m \\ 1 & \cdots & 1 & 1 \end{bmatrix}$$

has rank $p + 2$ yielding a contradiction.///

2. Convex Sets

2.1 Definition: A set, C, is said to be convex if $x^1, x^2 \in C$ implies $\lambda x^1 + (1-\lambda)x^2 \in C$ for all $0 \leq \lambda \leq 1$.

2.2 Lemma: If C is convex and $x^1, \ldots, x^p \in C$ then $\sum \lambda_j x^j \in C$ if $\lambda_j \geq 0$, $\sum \lambda_j = 1$.

Proof: By induction on p. For $p \leq 2$ obvious. Suppose true for $p \leq P$. Then

$$\sum_{j=1}^{P-1} \frac{\lambda_j x^j}{\sum_{j=1}^{P-1} \lambda_j} \in C \qquad \text{and}$$

$$\sum_{j=1}^{P-1} \lambda_j \left[\sum_{j=1}^{P-1} \frac{\lambda_j x^j}{\sum_{j=1}^{P-1} \lambda_j} \right] + \left[1 - \sum_{j=1}^{P-1} \lambda_j \right] x^P$$

$$= \sum_{j=1}^{P} \lambda_j x^j \in C \;.///$$

The relative interior of a subset A of E^m is the interior of A relative to V_A the linear variety of lowest dimension containing A.

2.3 Theorem: A non-empty convex set, C, has relative interior points.

Proof: Let k = a.d.(C) and suppose $x^0, \ldots, x^k \in C$ and are affinely independent. Then the "simplex" $S = \{x \,/\, x = \sum \lambda_i x^i,\ \sum \lambda_i = 1,\ \lambda_i \geq 0\}$ contains an open

set in the relative topology and clearly $S \subset C$. To see that S has an interior, let $y = \sum \frac{1}{k+1} x^i$. For any sequence of points in V, $y^j \to y$ such that

$$y^j = \sum \lambda_i^j x^i, \quad \sum_i \lambda_i^j = 1 \qquad j = 1,2,\ldots,$$

we have by continuity that for some J, $j \geq J$ implies $\lambda_i^j > 0 \qquad i = 0,\ldots,k.$///

2.4 Definitions: $C \subset E^m$ is a <u>cone with vertex at the origin</u> if $0 \in C$ and if $x \in C$ then $\lambda x \in C$ for all $\lambda \geq 0$. Any affine transformation of a cone-with-vertex-at-the-origin is simply called a <u>cone</u>. A <u>ray</u>, $R(x,y)$, is a cone of the form $R(x,y) = \{(1-\lambda)x + \lambda y / \lambda \geq 0\}$ for some fixed x and y, $x \neq y$.

$H = \{x / ax = b,\ a \neq 0,\ x \in E^m\}$ is called a <u>hyperplane</u> in E^m and $S = \{x / ax \geq b,\ a \neq 0,\ x \in E^m\}$ is called a <u>closed half-space</u> or for our purposes simply a half-space. If for any set $K \subset E^m$ we have $ax \geq b$ for all $x \in K$ we say that H(S) is a <u>bounding hyperplane (half-space)</u> of K. If moreover there exists in K a point x such that $ax = b$, then H(S) is said to be a <u>supporting hyperplane (half-space)</u>.

2.5 <u>Lemma</u>: <u>Half-spaces and hyperplanes are convex</u>.

<u>Proof</u>: Obvious.

2.6 <u>Lemma</u>: <u>The intersection of arbitrarily many convex sets is convex</u>.

<u>Proof</u>: Obvious.

2.7 Definition: The intersection of a finite number of half-spaces, K, is called a convex polyhedral set.

By 2.6, a convex polyhedral set is convex.

2.8 Definitions: The intersection, H(S), of all convex sets containing a set S is called the convex hull of H(S).

If $y = \sum_{j=1}^{n} \lambda_j x^j, \ \lambda_j \geq 0, \ \sum \lambda_j = 1$ then y is said to be a non-negative centroid of the x^j $j = 1, \ldots, n$.

2.9 Lemma: H(S), the convex hull of a set S, is equal to S', the set of all non-negative centroids of points in S.

Proof: Clearly $S \subset S'$ since $x = 1 \cdot x \in S$ for $x \in S$. On the other hand, $S' \subset H(S)$ since

$$y = \sum \lambda_j x^j, \quad \lambda_j \geq 0, \quad \sum \lambda_j = 1$$ belongs to every convex set containing S by an argument similar to the one in 1.6.///

We give now a few facts from the theory of convex polyhedral sets without proof since that would take us too far afield.

2.10 Theorem: A bounded set S is a convex polyhedral set iff it is the convex hull of a finite number of extreme points.

Proof: [8].

2.11 Definition: A point x^0 belonging to a convex set, C, is an extreme point of C if

$x^0 = \lambda_0 x^0 + \sum \lambda_j x^j, \quad \lambda_j \geq 0, \quad x^j \epsilon C, \; x^j \neq x^0 \; \sum \lambda_j = 1$ implies that $\lambda_0 = 1, \quad \lambda_j = 0, \quad j \neq 0.$

2.12 Theorem: Let $C \subset E^m$ be a closed convex cone with vertex at the origin. If d_1 is the dimension of the largest subspace contained in C, and d_2 is the dimension of the smallest subspace containing $H = \{\pi / \pi x \geq 0$ for $x \epsilon C$ then $d_1 + d_2 = m$.

Proof: [6 : p.12]

2.13 Theorem: If P is a convex polyhedral set and x^0 is an extreme point of P, then the cone $C(x^0) = \{\pi /$ there exists "a" such that $\pi x \geq a$ determines a support of P and $\pi x^0 = a\}$ has a non-empty interior.

Proof: Consider $P' = \{y - x^0 / y \epsilon P\}$. Let $C' = \{y / y = \lambda x, \lambda \geq 0, x \epsilon P\}$ $\pi x \geq 0$ for all $x \epsilon P$ iff $\pi x \geq 0$ for all $x \epsilon C'$. Clearly 0 is an extreme point of P therefore in theorem 2.12 we note that $d_1(C') = 0$. Hence $d_2(C) = m$. Theorem 2.3 completes the proof.///

Now returning to the question of general convex sets we prove:

2.14 Theorem: Let K be a closed, bounded, convex set and let $C = \{y / y = \lambda x, \lambda \geq 0, x \epsilon K\}$. C is closed.

Proof: Suppose $\lambda_j x^j$ converges to y where $\lambda_j \geq 0$, $x^j \epsilon K$, $j = 1, 2, \ldots.$ We must show that $y \epsilon C$. K is compact hence $\{x^j\}$ has a limit point x. Consider a subsequence x^{j_1}

$i = 1,2,\ldots$ which converges to x, and the sequence $\lambda_{j_i} x$. Since x^j are bounded λ_j are bounded and λ_{j_i} has a limit call it λ. Clearly $\lambda \geq 0$ and $y = \lambda x$.///

2.15 Definition: Suppose K is a closed convex set. Let x be any point of K and let $C = \{y - x \,/\, R(x,y) \epsilon K\}$ we call C the characteristic cone of K. $R(x,y)$ is the ray with origin x going through y. The definition is justified by the following theorem.

2.16 Theorem: The set $C(x) = \{y - x \,/\, R(x,y) \epsilon K\}$ is independent of x, $x \epsilon K$, K closed and convex.

Proof: First we show that $R(x,y) \subset K$ iff there exist $x \epsilon K$, $y^j \epsilon K$ such that $\|y^j\| \to \infty$ and limit

$$\frac{y^j - x}{\|y^j - x\|} = \frac{y - x}{\|y - x\|}.$$ Letting $y^j = x + j(y - x)$ we obtain the only if. Suppose $\lim \frac{y^j - x}{\|y^j - x\|} = \frac{y - x}{\|y - x\|}$ where $y^j \epsilon K$, $\|y^j\| \to \infty$. Let $Z \epsilon R(x,y)$, then $Z = x + \lambda_0 \frac{(y - x)}{\|y - x\|}$ for some $\lambda_0 \geqslant 0$. For $\lambda \leq \|y^j - x\|$

we have that $Z_j(\lambda) = x + \lambda \frac{(y^j - x)}{\|y^j - x\|} \epsilon K$. But $Z - Z_j(\lambda_0) =$

$$\lambda_0 \left(\frac{y - x}{\|y - x\|} - \frac{y^j - x}{\|y^j - x\|}\right).$$ Hence $Z_j(\lambda_0)$ approaches Z

and since K is closed $Z \epsilon K$ and since Z_0 was arbitrarily chosen, $R(x,y) \subset K$. But the limit $\frac{y^j - x}{\|y^j - x\|} = \lim \frac{-y^j}{\|y^j\|}$ is is independent of x.///

2.17 Lemma: Let K be a closed convex set and $y \notin K$.

Then there exists π such that

$\inf\{(\pi, x) \; / \; x \in K\} < \pi y.$

Proof: The euclidean norm $\| y - x \|$ is a continuous function of x hence it attains its minimum in K, at say x^0. Clearly x^0 is a boundary point of K. Let $\pi = x^0 - y$. $\pi y = -(x^0 - y)(x^0 - y) + \pi x^0 < \pi x^0$ since $y \neq x^0$. On the other hand if $x \epsilon K$, then $x^0 + \lambda(x - x^0) \epsilon K$ for $0 \leq \lambda \leq 1$. By the definition of x^0 we have $\| y - x^0 \| \leq \| (y - x^0) - \lambda(x - x^0) \|$ for $0 \leq \lambda \leq 1$. Squaring we obtain after cancellations and dropping the common factor $\lambda > 0$, $0 \leq 2(y - x^0)(x - x^0) - \lambda(x - x_0)^2$ letting $\lambda \to 0$ we have $(x^0 - y)(x - x^0) \geq 0$ hence $\pi x \geq \pi x^0 > \pi y$.///

2.18 Theorem: Let K be a convex set and y a boundary point then there exists π such that $\pi y = \inf\{\pi x \; / \; x \epsilon K\}$, i.e., through every boundary point of K there passes a supporting hyperplane.

Proof: Let y^j approach y such that $y^j \notin K$. Then there exists π^j satisfying Lemma 2.17. We can assume without loss of generality that $\| \pi^j \| = 1$ hence there exists π which is an accumulation point of the π^j. Since $\pi^j(x - y^j) > 0$ for all j and $x \epsilon K$ we have $\pi(x - y) \geq 0$.///

B. Ordinary Differential Equations:

0. Introduction: Contained in this appendix are classical results from the theory of ordinary differential equations. References will be given where the proofs of these results can be found.

1. General Theory: When confronted with a differential equation $\frac{dx}{dt} = f(x,t)$ one often considers the related integral equation $x(t) = x(0) + \int_0^t f(x,t)dt$. The first equation implies that $\frac{dx}{dt}$ exists at every point of interest while the Lebesgue theory of integration merely guarantees that solutions of the integral equation are absolutely continuous. To make the identification more complete, it is convenient to interpret $\frac{dx}{dt} = f(x,t)$ as holding only almost everywhere (which we abbreviate a.e.). In particular we will study the following problem:

Find an absolutely continuous function $x(t)$ defined on $I = [t_1, t_2]$ such that

$$(x(t),t) \in \mathcal{B} \subset E^n \times T$$

$$\frac{dx}{dt} = f(x(t),t) \text{ a.e. on } I, \tag{1.1}$$

and $\quad x(\tau) = \xi$ for some fixed $\tau \in I$.

1.2 Remark: Since $x(t)$ is absolutely continuous, it is differentiable a.e. [9 ; pp.203-205].

1.3 Theorem: Let f be defined on

$R[a,b,t_1,t_2] = \{x_1,\ldots,x_n,t \;/\; a_i \le x_i \le b_i,\ i = 1,\ldots,n,$

$t_1 \le t \le t_2\}$ where $a_i < b_i$, $i = 1, \ldots, n$ and $t_1 < t_2$ and suppose it is measurable in t for each fixed x and continuous in x for each fixed t. If there exists a Lebesgue-integrable function m(t) on the interval $t_1 \le t \le t_2$ such that $||f(x,t)|| \le m(t)$ for $(x,t) \varepsilon R[a,b,t_1,t_2]$ then there exists a solution, x(t), to (1.1) on some interval $t_1 \le t_1' \le t \le t_2' \le t_2$ satisfying $x(\tau) = \xi$, for $a_i \le \xi_i \le b_i$.

1.4 Theorem: In an open connected domain $\mathcal{D} \subset E^n \times T$, let f be defined, measurable in t for fixed x and continuous in x for fixed t. Let m be an integrable function such that $||f(x,t)|| \le m(t)$ for $(x,t) \varepsilon \mathcal{D}$. Then given a solution, x(t), to (1.1) on a non-empty interval $t_1 < t < t_2$, it can be extended to the boundary of $\mathcal{D}$.

1.5 Definitions: A function $\phi(x)$ is said to satisfy a Lipshitz condition with constant K on the domain D if $|| \phi(x_1) - \phi(x_2)|| \le K||x_1 - x_2||$ for all $x_1, x_2 \varepsilon D$. $\phi(x)$ is said to satisfy a Lipshitz condition locally if, on every neighborhood in D, $\phi(x)$ satisfies a Lipshitz condition for some constant K which may depend on the neighborhood.

1.6 Theorem: If, in addition to the hypotheses of 1.3, f also satisfies a Lipshitz condition in x locally, then the solution is unique.

1.7 Theorem: Suppose $f = f(x,t,\mu)$ is a function of $x \varepsilon E^n$, $t \varepsilon T$, and $\mu \varepsilon E^m$ on a domain $\mathcal{D}_\mu = \mathcal{D} x\ I_\mu$ where $I_\mu = \{\mu / \ |\mu - \mu_0| < C\}$, $C > 0$. Suppose f is measurable in t

for fixed μ and x; continuous in x for fixed t and μ and for fixed t, f is continuous in (x,μ) at $\mu = \mu_0$. Moreover we assume there exists a Lebesgue integrable function m(t), for $t_1 \leq t \leq t_2$, such that $\| f(x,t,\mu) \| \leq m(t)$, uniformly in x,μ for $t,x,\mu \in \mathcal{D}_\mu$. Then if for $\mu = \mu_0$

$$\frac{dx}{dt} = f(x,t,\mu_0), \quad x(\tau) = \xi$$

has a unique solution on $[t_1,t_2]$ where $\tau \in [t_1,t_2]$ there exists $\delta > 0$ such that for any fixed (σ,η,μ) satisfying

$$| \sigma - \tau | + \| \eta - \xi \| + \| \mu - \mu_0 \| < \delta,$$

all solutions $x(t,\sigma,\eta,\mu)$ of

$$\frac{dx}{dt} = f(x,t,\mu), \quad x(\sigma) = \eta$$

exist on $[t_1,t_2]$. Moreover as $(\sigma,\eta,\mu) \to (\tau,\xi,\mu_0)$ $x(t,\sigma,\eta,\mu) \to x(t,\tau,\xi,\mu_0)$ uniformly on $[t_1,t_2]$.

Proofs: Proofs of theorems 1.3, 1.4, 1.6, and 1.7 can be found in [1].

2. Linear Differential Equations: In this section we apply the theorems of the previous section to linear differential equations and, in addition, obtain some special results.

First we consider the homogeneous linear differential equation:

(2.1) $\frac{dy(t)}{dt} = A(t)\, y(t)$, where $A(t)$ is a $n \times n$ matrix of functions of t. If $\| A(t) \| \leq M$ on some time interval $[t_1,t_2]$, then $A(t)\, y(t)$ satisfies a Lipshitz condition on $I = [t_1,t_2]$ with constant M. Applying theorems 1.3, 1.4 and 1.6 we obtain:

2.2 Theorem: If each element of A is measurable on $I = [t_1,t_2]$ and $\| A(t) \| \leq m(t)$ on I where $m(t)$ is Lebesgue integrable then for any $\xi \epsilon E^n$ there exists a unique solution, $y(t)$, to (2.1) (i.e., (2.1) holds almost everywhere) with $y(t) = \xi$ on the interval I.

Obviously $y(t) \equiv 0$ is a solution of (2.1), so by uniqueness if $y(t) = 0$ for any $t \epsilon I$, $y(t) \equiv 0$ on I. Let $y^j(t)$ be the solutions of (2.1) with initial conditions $y^j(\tau) = e^j$ where $e^1,\ldots,e^n$ is any basis for E^n. Let $y(t)$ be any solution of (2.1). Suppose $y(\tau) = \xi$. By the definition of $e^1,\ldots,e^n$ there exists $\lambda_1,\ldots,\lambda_n$ such that $\xi = \sum \lambda_j e^i$. If we let $Z = y(t) - \sum \lambda_j y^j(t)$ we have $Z(\tau) = 0$, hence since $Z(t)$ is obviously a solution of (2.1), $Z(t) \equiv 0$ on I. This essentially proves:

2.3 Theorem: The solutions of (2.1) in I form a vector space (of functions) of dimension precisely n.

Now we turn to the non-homogeneous problem.

(2.4) $\frac{dx}{dt} = A(t)\ x(t) + b(t)$ a.e.,

where $A(t)$ is a $n \times n$ matrix of functions and $b(t)$ is a n-vector of functions. In this case we can exhibit a solution to (2.4) if we know a basis of solutions to the "reduced system" (2.1).

2.5 Theorem: If $A(t)$ and $b(t)$ are integrable functions of t on I and $\|A(t)\| \leq m(t)$, $\|b(t)\| \leq m(t)$, where $m(t)$ is Lebesgue integrable on I then for any point $\xi \epsilon E^n$ and $\tau \epsilon I$ (2.4) has a unique solution such that $x(\tau) = \xi$ of the form

(2.6) $x(t) = Y(t)\xi + \int_\tau^t Y(t)\ Y^{-1}(s)\ b(s)\ ds$

where $Y(t)$ is the matrix solution of

$$\frac{dY(t)}{dt} = A(t)\ Y(t)$$

$Y(\tau) = I$ the $n \times n$ identity matrix.

The proofs of theorems in this section can be found in [2 ; Ch. 3].

3. The variational equations associated with a system of ordinary differential equations and the corresponding adjoint system: The principal theorem we need here is the following:

3.1 Theorem: Let $f(x,t,\mu)$ satisfy: (a) For every fixed value μ in some neighborhood U_μ of the point μ_0 the components of $f(x,t,\mu)$ are measurable as functions of t for fixed x, continuous in x for fixed t satisfying

$t_1 \leq t \leq t_2$ and

$$\| f(x,t,\mu) \| \leq M(t,\mu) \quad \text{for } t_1 \leq t \leq t_2,\ \mu\epsilon U_\mu \text{ where}$$

$M(t,\mu)$ is integrable in t for fixed μ.

(b) Along a solution

$$x^0(t) = x(t,\mu_0),\ x^0(t_0) = x(t_0,\mu_0) = x^0(\mu_0)$$

$f(x,t,\mu)$ is differentiable in (x,μ).

(c) If we let

$$r = \sqrt{|h|^2 + |h'|^2}$$

then whenever $\mu_0 + h$ lies in U_μ we have

$$\left\| \frac{f(x+h',\ t,\ \mu_0+h) - f(x,t,\mu_0)}{r} \right\| \leq K(t)$$

where $K(t)$ is a summable function on $t_1 \leq t \leq t_2$.

Then if the solutions of

(3.2) $$\frac{dx}{dt}(t,\mu) = f(x(t,\mu),\ t,\ \mu)$$

$$x(t_0;\mu) = x^0(\mu)$$

are differentiable in μ at the point $t = t_0$, $\mu = \mu_0$, they are differentiable in μ for arbitrary values of t in the

region $t_1 < t < t_2$, $\mu = \mu_0$, and their derivatives with respect to μ satisfy the "variational equations":

(3.3) $\frac{dy}{dt} = \nabla_x f \cdot y + \frac{\partial f}{\partial \mu}$ where $\nabla_x f$ is the matrix of partial derivatives of $f = (\frac{\partial f^i}{\partial x_j})$ and $\nabla_x f$ and $\frac{\partial f}{\partial \mu}$ are evaluated along the solution $x(t,\mu_0)$.

(3.3) is called the variational equation associated with (3.2). An important special case arises when only the initial point $x^0(\mu)$ depends on μ yielding the homogeneous variational equation

(3.4) $\frac{dy}{dt} = \nabla_x f \Big|_{x(t) = x(t,\mu_0)} \cdot y$

Since we will have no occasion to use (3.3) we simply call (3.4) the variational equation associated with (3.2).

Solutions of (3.4) can be interpreted as the incremental change to the trajectory $x(t,\mu_0)$ which results from changing the initial point from $x^0(\mu_0)$ to $x^0(\mu)$. Formally, (3.3) and (3.4) can be derived by using the uniformly continuous dependence of the solution to differential equations on the initial values and f to make a first order approximation to the incremental change in the trajectory (Theorem 1.7). Then if f is expanded in a Taylor series with respect to x and μ, the desired equations can be obtained (although stronger assumptions are needed on f for this approach).

The extension of Theorem 3.1 to the case where μ is a vector can also be carried out in which case (3.3) becomes

(3.5) $\frac{dy}{dt} = \nabla_x f\ y + \nabla_u f.$

Now let us return to variations only in the initial point. We take $\mu \epsilon E^n$ and

$x^0(\mu) = (x_1^0 + \mu_1, \ldots, x_n^0 + \mu_n)$ and $\mu_0 = 0$. Then for each t, $t_1 < t < t_2$, $y(t;\mu)$ defines through (3.4) a linear homogeneous transformation of the n dimensional μ-space (or equivalently the y space at time t_0) onto the n-dimensional y-space at time t. Let a hyperplane in the μ space passing through the origin be determined by

$$\pi(t_0)\mu = 0 \quad \text{or equivalently} \quad \pi(t_0)\, y\, (t_0, \mu) = 0$$

This hyperplane is transformed at time t to a new hyperplane determined by $\pi(t)\, y\, (t;\mu) = 0$. To find $\pi(t)$ it suffices to find $\pi(t)$ such that $\pi(t)\, y\, (t,\mu)$ is constant in time for each fixed μ. For this to occur, the following must be satisfied:

$$(3.5) \quad 0 = \frac{d}{dt}\, \pi(t)\, y\, (t,\mu) = \dot{\pi}(t)\, y\, (t,\mu) + \pi(t)\, \dot{y}\, (t,\mu)$$
$$= \dot{\pi}(t)\, y\, (t,\mu) + \pi(t)(\nabla_x f)\, y\, (t,\mu)$$

The above equation can be inverted (Theorem 2.3) by choosing μ^j, $j = 1, \ldots, n$ linearly independent. Thus if

(3.6) $\frac{d\pi}{dt} = -\pi(t)\ \nabla_x f$ for $\nabla_x f$ evaluated along the solution $x(t,0)$ is satisfied then so is (3.5). (3.6) is called the <u>adjoint equation</u> for (3.4).

The proofs required for this section can be found in [1].

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Listed here are only the works referred to in the text.

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